TAKING LIMITS UNDER THE INTEGRAL SIGN

F. CUNNINGHAM, JR., Bryn Mawr College

1. Introduction. This expository paper is addressed to undergraduates for whom "definite integral" means the integral defined by Riemann, and who need occasionally to take limits under the integral sign. This useful manipulation, also called in the context of infinite series "integration term by term," consists in reversing the order of two analytic operations, one being definite integration with respect to one variable, the other taking the limit with respect to another variable. To fix the notation consider one representative case, that of a sequence $f_1, f_2, \cdots$ of functions, each of which is integrable on the same interval $[a, b]$, and such that for each x in $[a, b]$ $\lim_n f_n(x) = f(x)$ exists. We wish to conclude that $\int_a^b f = \lim_n \int_a^b f_n$. It is unfortunately possible for either side of this equation, or both, not to exist, or even for both to exist and be unequal. Examples are given in Advanced Calculus, and a typical one will be found in Section 2 below. This being so, each time this manipulation is used a reason must be given for its correctness in that case. The commonly taught tool for this purpose is the following theorem, which dates from the mid-nineteenth century.

UNIFORM CONVERGENCE THEOREM. *If the sequence $\{f_n\}$ of integrable functions converges uniformly on $[a, b]$ to f, then f is integrable and $\int_a^b f = \lim \int_a^b f_n$.*

Integration theory has value only to the extent that we know some functions which are integrable, i.e., whose integrals exist. Two useful criteria for integrability are that the function be continuous, or that it be piecewise monotone, either of these conditions being sufficient. Half of the above theorem is in effect a third guarantee of integrability. It often happens in applications, however, that the limit function f is known by an explicit formula, is seen to be continuous or piecewise monotone, so that its integrability is known in advance. In that case a much weaker hypothesis than uniform convergence is enough to insure that $\int_a^b f = \lim \int_a^b f_n$. Namely, it is enough to have all the functions bounded by the same constant (see Theorem 1 below), or even less than this (see Theorem 2). Knowing one of these theorems not only obviously increases one's power to take limits in certain cases, and relieves one in other cases of the burden of an irrelevant test for uniform convergence; it also gives one a truer understanding of what makes term-by-term integration give the right answer.

The discovery by C. Arzelà of the bounded convergence theorem, as Theorem 1 is called, preceded by a few years the famous work of Lebesgue at the beginning of this century, which carried the matter farther. By refining the definition of the integral, Lebesgue extended integration to many functions which are too discontinuous to have Riemann integrals, in such a way that the limit function in the bounded (or dominated) convergence theorem is automatically integrable. This made the Riemann integral obsolete. It also eliminated the role of uniform convergence in integration theory, which was only to carry the integrability of the functions in a sequence over to the limit. Unfortunately Lebesgue integration is more complicated than Riemann integration, and this puts Lebesgue's proofs of the theorems in this paper out of reach of the readers to whom this is addressed. References to Arzelà and to later simplifications of his proof by other authors can be found in [1]. The method used here has the virtue of requiring a minimum of previous knowledge. Only the most basic notions of real analysis are used up to Section 4, where the subject takes a slightly more technical turn.

2. The bounded convergence theorem. In order to present the argument in its simplest form, I begin with a theorem which, while good enough for many applications, is not the most general possible. Improvements are discussed in Section 4. As usual, inequalities involving functions asserted to hold on a set A mean that the corresponding inequalities between values of the functions hold for all points in A. When no set A is mentioned, the inequality applies to the whole domain, which is always $[a, b]$. I write $\int$ for $\int_a^b$ and $|f|$ for the function whose value at x is $|f(x)|$.

THEOREM 1. *Let $\{f_n\}$ be a sequence of integrable functions defined on $[a, b]$. Assume*

(i) *For each x in $[a, b]$ $f_n(x)\to f(x)$ (as $n\to\infty$), where f is integrable on $[a, b]$.*

(ii) *There exists a constant K such that $|f_n|\leqq K$ for all n. Then $\int f_n\to\int f$.*

Examples of the following kind help to illuminate the force of hypothesis (ii). Take $[a, b]$ to be $[0, 1]$. Let f be any nonnegative integrable function defined on $[0, 1]$, with $f(0)=0$ and $\int_0^1 f=J\neq 0$; for instance $f(x)=\sin \pi x$. Extend f to $[0, \infty)$ by setting $f(x)=0$ when $x>1$, and set $f_n(x)=c_nf(nx)$, where $c_1, c_2, \cdots$ are adjustable constants. (The graph of f_n has the same shape as that of f, but compressed into the interval $[0, 1/n]$ and with height changed by the factor c_n.) Regardless how the c_n are chosen, $f_n(x)\to 0$ for each x in $[0, 1]$. The convergence to 0 is uniform if and only if $c_n\to 0$. Hypothesis (ii) is satisfied if and only if the sequence $\{c_n\}$ is bounded. Thus Theorem 1 applies, while the uniform convergence theorem does not, if for instance $c_n=1$ for all n, and the conclusion of the theorem is easily checked by direct calculation: $\int_0^1 f_n=c_nn^{-1}J\to 0$. By contrast, taking $c_n=n$ in violation of (ii) gives $\int_0^1 f_n=J$. Since $\lim J=J\neq 0$, the conclusion of the theorem is in this case false. (The gap between $\{c_n\}$ bounded and $c_n=n$ is a weakness of the theorem to be remedied later.)

What the theorem is saying is: if you squeeze a tube of toothpaste towards flatness at every point, and if there is a bound on the cross-section of all bulges, then all the toothpaste must come out.

The heart of Theorem 1 is contained in a lemma of Arzelà, which is actually a very special case of the theorem. I shall prove the theorem from the lemma in this section, and in the next section do the interesting part, which is the proof of the lemma. The lemma has to do with subsets of $[a, b]$ of a particularly simple kind, namely finite unions of closed intervals. Call this family of sets $\mathfrak{F}$. For $F\in\mathfrak{F}$ I write $|F|$ to mean the *measure* of F, that is the sum of the lengths of the disjoint closed intervals whose union is F.

LEMMA 1. (Arzelà). *Let $\{F_n\}$ be an infinite sequence of sets belonging to $\mathfrak{F}$. Assume there exists $\epsilon>0$ such that $|F_n|\geqq\epsilon$ for all n. Then there is some x in $[a, b]$ which belongs to infinitely many F_n.*

To see that this is a special case of Theorem 1, let f_n be, for each n, the characteristic function of F_n, equal to 1 on F_n and 0 everywhere else. Then f_n is integrable; in fact $\int f_n = |F_n|$. To say for some x in $[a, b]$ that $f_n(x) \to 0$ is to say that $x \in F_n$ for only a finite number of n. Thus if we assume the conclusion of the lemma false, then $\{f_n\}$ satisfies (i) of Theorem 1. Since (ii) is trivial for characteristic functions, it would follow from Theorem 1 that $|F_n| \to 0$, contradicting $|F_n| \geqq \epsilon$.

The proof that, conversely, Lemma 1 implies the theorem is as follows: For each n set $g_n = |f_n - f|$. Then $g_n \geqq 0$, $g_n(x) \to 0$ for all x, and $g_n \leqq 2K$. If the theorem were false, then we would have an $\epsilon > 0$ such that $\int g_n \geqq \epsilon$ for infinitely many n. For each of these n apply Lemma 2 below to g_n to obtain a set $F_n \in \mathfrak{F}$ such that $g_n \geqq \epsilon/4(b-a)$ on F_n and $|F_n| \geqq \epsilon/8K$. Then by Arzelà's Lemma some x in $[a, b]$ belongs to infinitely many F_n. For this x we have $g_n(x) \geqq \epsilon/4(b-a)$ infinitely many times, contradicting $g_n(x) \to 0$, and finishing the proof.

LEMMA 2. *Let g be a nonnegative integrable function on $[a, b]$ satisfying $g \leqq K'$ and $\int g \geqq \epsilon$. Then there exists a set F belonging to $\mathfrak{F}$ such that $|F| \geqq \epsilon/4K'$ and $g \geqq \epsilon/4(b-a)$ on F.*

Proof. I shall call a *lower sum* for g any number expressible in the form $\sum_{i=1}^{r} y_i |I_i|$ where $I_1, \cdots, I_r$ are closed intervals, disjoint except for common endpoints, whose union is $[a, b]$ (briefly, a *partition* of $[a, b]$), and $y_1, \cdots, y_r$ are numbers such that $g \geqq y_i$ on I_i for $i = 1, \cdots, r$. By definition $\int g$ is the least upper bound of all lower sums for g. Therefore, since $\int g > \frac{1}{2}\epsilon$, we can find $I_1, \cdots, I_r$ and $y_1, \cdots, y_r$ so that the lower sum $\sum_{i=1}^{r} y_i |I_i| > \frac{1}{2}\epsilon$. For F take the union of those I_i for which the corresponding $y_i \geqq \epsilon/4(b-a)$. Then certainly $g \geqq \epsilon/4(b-a)$ on F. To estimate $|F|$ it is convenient to write $\sum_F$ for sums over only those indices corresponding to intervals included in F, and $\sum_{F'}$ for sums over the remaining indices. Thus $|F| = \sum_F |I_i|$. Now for terms in $\sum_{F'}$, $y_i < \epsilon/4(b-a)$, so

$$\sum_{F'} y_i |I_i| < [\epsilon/4(b-a)] \sum_{F'} |I_i| \leqq \tfrac{1}{4}\epsilon.$$

Therefore, altogether,

$$\tfrac{1}{2}\epsilon < \sum_{i=1}^{r} y_i |I_i| = \sum_{F} y_i |I_i| + \sum_{F'} y_i |I_i| \leqq K' |F| + \tfrac{1}{4}\epsilon$$

from which follows $|F| \geqq \epsilon/4K'$.

3. Arzelà's Lemma. Lemma 2 has an intuitively digestible content. It is a "crowding principle" to the effect that too many large sets cannot fit into a finite space without much overlapping. It can be compared to the elementary combinatorial fact that if $A_1, \cdots, A_m$ are subsets of a finite set S having only N elements, and if each A_i has at least n elements, then some element of S must belong to at least mn/N of the sets A_i. In Arzelà's Lemma the cardinal numbers N and n are replaced by continuous size-measures, $b-a$ and ϵ respectively, and m is infinite.

To show how nontrivial Arzelà's Lemma is, be it remarked that its truth depends on the completeness of the real number system, and therefore so does Theorem 1. I mean this in the strong sense that a context can be envisaged in which these theorems make sense and are false, namely, replace the real interval $[a, b]$ by the rational interval $[a, b]$ consisting of the rational numbers between a and b inclusive. Indeed, for each positive integer n let R_n be the set of all those rational numbers in $[a, b]$ whose fractional representations in lowest terms have denominators $\geqq n$. Then each R_n contains all except a finite number of the rationals in $[a, b]$, and it is easy to construct a set F_n belonging to the rational counterpart of $\mathfrak{F}$, contained in R_n, and satisfying $|F_n| \geqq \frac{1}{2}$. Now any rational number m/k in lowest terms belongs to R_n, and therefore also F_n, only at most for $n=1, 2, \cdots, k$, that is, for a finite number of n. Thus for the rational number system as domain, the sequence $\{F_n\}$ is a counterexample to Lemma 2, and the corresponding sequence of characteristic functions is a counterexample to Theorem 1.

The example just given exhibits the topological ingredient of the theorem. There is also a combinatorial ingredient, which is isolated in the next lemma. The relevant properties of $\mathfrak{F}$ and measure used here are only that $\mathfrak{F}$ is closed under the taking of finite unions and intersections (unfortunately not also under the taking of complements), and two axioms for measure:

1. Monotonicity: If $E_1 \subset E_2$, then $|E_1| \leqq |E_2|$
2. Additivity: $|\phi| = 0$, and for all E_1 and E_2 in $\mathfrak{F}$

$$|E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2|$$

If these facts are not considered obvious, they are most easily proved by temporarily enlarging $\mathfrak{F}$ to include all finite unions of intervals (closed or not), so that relative complementation becomes possible in $\mathfrak{F}$. The definition of measure applies equally well to sets in this larger family, and Axiom 2 follows by use of relative complements from its less general form

2′. If $E_1 \cap E_2 = \phi$, then $|E_1 \cup E_2| = |E_1| + |E_2|$

LEMMA 3. *Let $G_0 \in \mathfrak{F}$, and let $\{F_n\}$ be a sequence of sets in $\mathfrak{F}$ contained in G_0 such that $|F_n| \geqq \epsilon$ for all n, where $\epsilon > 0$. Then there exists a nested sequence $G_0 \supset G_1 \supset G_2 \cdots$ of sets in $\mathfrak{F}$ such that $|G_n| \geqq \frac{1}{2}\epsilon$ for all n and every point of $G = \bigcap_{n=1}^{\infty} G_n$ belongs to infinitely many F_n.*

Proof. Choose a sequence $\delta_1, \delta_2, \cdots$ of numbers such that $\epsilon > \delta_1 > \delta_2 > \cdots > \frac{1}{2}\epsilon$; for instance $\delta_n = (\frac{1}{2} + \frac{1}{4}n^{-1})\epsilon$ would do. Now $|F_1 \cup \cdots \cup F_n|$ increases with n and is bounded above by $|G_0|$ (Axiom 1), so converges to some limit L_1. Let $G_1 = F_1 \cup \cdots \cup F_{n_1}$, where n_1 is chosen so large that $|G_1| > L_1 - (\epsilon - \delta_1)$. Then for all $n > n_1$ we have $|G_1 \cup F_n| \leqq |F_1 \cup \cdots \cup F_n| \leqq L_1 < G_1 + \epsilon - \delta_1$. Hence (Axiom 2)

$$|G_1 \cap F_n| = |G_1| + |F_n| - |G_1 \cup F_n| \geqq |G_1| + \epsilon - |G_1| + \epsilon - \delta_1) = \delta_1.$$

We can now repeat the construction, using the sequence of sets $G_1 \cap F_{n_1+1}$, $G_1 \cap F_{n_1+2}, \cdots$, and with ϵ replaced by δ_1. Namely $|G_1 \cap (F_{n_1+1} \cup \cdots \cup F_{n_1+n})|$

converges to some limit L_2. We define $G_2=G_1\cap(F_{n_1+1}\cup\cdots\cup F_{n_2})$ taking n_2 so large that $|G_2|>L_2-(\delta_1-\delta_2)$. Then we conclude as before that for $n>n_2$ $|G_2\cap F_n|\geqq\delta_2$. In general, when we have found G_k and n_k, the construction gives an integer $n_{k+1}>n_k$ and a set $G_{k+1}=G_k\cap(F_{n_k+1}\cup\cdots\cup F_{n_{k+1}})$ belonging to $\mathfrak{F}$, contained in G_k, and such that $|G_{k+1}\cap F_n|\geqq\delta_{k+1}$ for all $n>n_{k+1}$. Now we check the other properties required in the lemma. Clearly $|G_k|\geqq\delta_k>\frac{1}{2}\epsilon$. Also, since $G_k\subset F_{n_{k-1}}+1\cup\cdots\cup F_{n_k}$, every point in G_k belongs to F_n for some n in the range $n_{k-1}<n\leqq n_k$. If $x\in G=\bigcap_{k=1}^{\infty}G_k$, this is true for every k, and therefore x belongs to F_n for infinitely many different n. Thus Lemma 3 is proved.

To finish the proof of Arzelà's Lemma, we apply Lemma 3 with $[a, b]$ for G_0; we then only need to make sure that G is not empty. But this, the topological ingredient, amounts to the compactness of the interval $[a, b]$, G being the intersection of a nested sequence of nonempty (see their measures) closed sets. For completeness I include a quick proof of the famous theorem used here. Let $x_n=\inf G_n$. It exists and belongs to G_n because G_n is nonempty, bounded, and closed. The sequence $\{x_n\}$ is increasing, because the sequence $\{G_n\}$ is nested, and therefore, being bounded above by b, converges to a limit x. For any n and all $k>n$ we have $x_k\in G_k\subset G_n$; whence $x\in G_n$, because G_n is closed. Since this applies to all n, $x\in G$.

4. The dominated convergence theorem. Both hypotheses of Theorem 1 are more restrictive than necessary, but the statement of a more general theorem involves additional technicalities. The aim of this section is to evolve Theorem 2 below by making the necessary small changes in the proof of Theorem 1.

To start with hypothesis (i), it can be relaxed so as to permit a set, provided it is not too large, of exceptions to the rule of pointwise convergence. A set S is said to be *of measure zero* if for every $\epsilon>0$ there exist open intervals $I_1, I_2, \cdots$ (finitely or infinitely many) covering S such that $\sum_{n=1}^{\infty}|I_n|<\epsilon$. For example, all finite sets and all denumerably infinite sets are of measure zero. Because of the Heine-Borel Theorem, any closed bounded set of measure zero can be covered by a finite number of open intervals $I_1, \cdots, I_N$ with $\sum_{n=1}^{N}|I_n|<\epsilon$.

The phrase "almost everywhere" is commonly used as an abbreviation for "with the exception of a set of measure zero."

As the first step in proving Theorem 2 let us show that Theorem 1 remains valid when hypothesis (i) is replaced by

(i′) $f_n(x)\to f(x)$ almost everywhere in $[a, b]$, f being integrable.

The proof of Theorem 1 clearly applies equally to this new version if we obtain in Arzelà's Lemma the stronger conclusion:

The set of x in $[a, b]$ belonging to infinitely many F_n is not of measure zero.

To prove this, use Lemma 3 as before, and then show that G is not only not empty, but not even of measure zero. (It was from foresight of this strengthening that the residual $\frac{1}{2}\epsilon$ was saved in Lemma 2.) Indeed, suppose G is of measure zero. Since G is closed and bounded we can find a finite number of open intervals $I_1, \cdots, I_N$ covering G with $\sum_{n=1}^{N}|I_n|<\frac{1}{4}\epsilon$. The complement of $I_1\cup\cdots\cup I_N$ relative to $[a, b]$ is a set H belonging to $\mathfrak{F}$, and $|H|>b-a-\frac{1}{4}\epsilon$. Now for all k,

$|G_k \cap H| = |G_k| + |H| - |G_k \cup H| > \frac{1}{2}\epsilon + b - a - \frac{1}{4}\epsilon - (b-a) = \frac{1}{4}\epsilon$. Thus $\{G_k \cap H\}$ is a nested sequence of nonempty closed sets, so $G \cap H = \bigcap_{k=1} G_k \cap H$ is not empty. But this contradicts the fact that G is covered by $I_1, \cdots, I_N$ and hence G is not of measure zero.

Turning now to hypothesis (ii) of Theorem 1, we shall see that it can be relaxed so as to permit in place of the constant bound K for all the f_n a (possibly unbounded) dominating function k, provided k is not too large. As a measure of the size of k, I shall use its lower integral, defined in terms of lower sums in the same way as the integral, but without requiring that the integral exist. (The lower integral is the Riemann integral only if it has the same value as the upper integral defined similarly by approximation from above. A function which is unbounded above has no upper sums and hence no upper integral.) For any nonnegative function g, the class of lower sums of g is not empty since it contains 0. If this set is bounded above we define the lower integral $\underline{\int} g$ of g on $[a, b]$ to be its least upper bound. Otherwise we write $\underline{\int} g = \infty$.

THEOREM 2. *Let f_n be a sequence of integrable functions on $[a, b]$. Assume*
(i′) *$f_n(x) \to f(x)$ almost everywhere in $[a, b]$, f being integrable.*
(ii′) *There exists a nonnegative function k such that $\underline{\int} k < \infty$ and $|f_n| \leq k$ for all n. Then $\int f_n \to \int f$.*

When applying this theorem it is up to the user to produce a suitable function k. If a *bounded* k exists which dominates all the f_n then the extra generality of (ii′) over (ii) is not needed, because k can be replaced by its constant upper bound. To exploit the generality of Theorem 2 one needs a supply of unbounded functions with finite lower integrals. Fortunately there are many. Suppose, for instance, that $\lim_{x \to a+} k(x) = +\infty$, that k is integrable on $[a+\epsilon, b]$ for every $\epsilon > 0$, and that $\int_a^b k$ is a convergent improper integral. Then it is easy to show that $\underline{\int} k \leq \int_a^b k < \infty$. (In fact the two integrals are equal.)

Example. Returning to the example in Section 2, take $c_n = n^p$ for an exponent $p > 0$. Then $\{f_n\}$ is not uniformly bounded, and Theorem 1 does not apply. But if K is an upper bound for f on $[0, 1]$ then, taking into account that $f_n(x) = 0$ for $x \geq 1/n$, it is clear that $f_n(x) \leq Kx^{-p} = k(x)$. Now $\underline{\int}_0^1 k = \int_0^1 Kx^{-p} dx < \infty$ if and only if $p < 1$, and this is precisely the condition under which $\int_0^1 f_n \to 0$, as is verified by direct calculation.

The proof of Theorem 2 imitates that of Theorem 1, but requires an extra step. After defining $g_n = |f_n - f|$ as before and, assuming the conclusion false, taking ϵ appropriately, one cannot immediately apply Lemma 2, because the functions g_n are perhaps not bounded. Note however that f, being integrable, is bounded so that by adding a constant if necessary we can assume $g_n \leq k$. The strategy is to cut the tops off the g_n to make a uniformly bounded sequence, being careful not to cut off so much as to lose the force of $\int g_n \geq \epsilon$ (for infinitely many n). The next lemma shows this is possible. I write $u \wedge v$ to mean the smaller, and $u \vee v$ to mean the larger of two numbers u and v. (If $u = v$, then $u \wedge v = u \vee v = u = v$.) Similarly $f \wedge g$ means the function whose value at x is always $f(x) \wedge g(x)$. Thus for a constant K, the truncated function $g \wedge K$ agrees with g wherever $g(x) \leqq K$ and has the value K everywhere else. The nonnegative function $g-g\wedge K$ represents the top of g which has been cut off at the level K.

LEMMA 4. *Let k be a nonnegative function with $\underline{\int} k<\infty$, and let $\epsilon>0$. Then there exists a constant K so large that for any nonnegative integrable function $g \leqq k$ $\int(g-g\wedge K)<\epsilon$.*

Proof. Let $\sum_{i=1}^{r} y_i|I_i|$ be a lower sum for k such that $\sum_{i=1}^{r} y_i|I_i|>\underline{\int} k-\epsilon$, and let K be the largest of the numbers $y_1, \cdots, y_r$ appearing in it. To show that this K works, let g be integrable, $0 \leqq g \leqq k$. If any lower sum of $g-g\wedge K$ is added to $\sum_{i=1}^{r} y_i|I_i|$, the resulting number is still a lower sum for k. This is because if $\Delta y_i \leqq g(x)-g(x)\wedge K$, then $y_i+\Delta y_i \leqq K+g(x)-g(x)\wedge K=g(x)\vee K \leqq k(x)$. Thus $\underline{\int} k-\sum_{i=1}^{r} y_i|I_i|$ is an upper bound for the lower sums of $g-g\wedge K$, and so is as big as their least upper bound, namely $\int(g-g\wedge K)$. Since $\underline{\int} k-\sum_{i=1}^{r} y_i|I_i|<\epsilon$, this proves the lemma.

To finish the proof of Theorem 2, apply Lemma 4 with $\frac{1}{2}\epsilon$ in place of ϵ to obtain K such that for all n $\int(g_n-g_n\wedge K)<\frac{1}{2}\epsilon$ and, hence, $\int(g_n\wedge K)>\int g_n-\frac{1}{2}\epsilon \geqq \frac{1}{2}\epsilon$. Now Theorem 1, in its strengthened form, applies to the uniformly bounded sequence $\{g_n\wedge K\}$ to give a set, not of measure zero, of points x in $[a, b]$ where $g_n(x)\wedge K$ does not converge to 0. For these x it is clear that also $g_n(x)$ does not converge to 0, so the theorem is proved.

5. Applications. I shall indicate only briefly some consequences of Theorems 1 and 2.

THEOREM 3. *Let $f(x, t)$ be a bounded function which is continuous in t for each x and integrable in x for each t. Then $F(t)=\int_a^b f(x, t)dx$ is continuous.*

Proof. To prove continuity at t_0 apply Theorem 1 to $f_n(x)=f(x, t_n)$ where $\{t_n\}$ is an arbitrary sequence converging to t_0.

THEOREM 4. *Let $f(x, t)$ be integrable in x for each t and have a bounded partial derivative $f_t(x, t)$ which is integrable in x for each t. Then $F'(t)=d/dt \int_a^b f(x, t)dx =\int_a^b f_t(x, t)dx$.*

Proof. Set $f_n(x)=n[f(x, t_0+n^{-1})-f(x, t_0)]$. Then (i) $f_n(x)\to f_t(x, t_0)$ for each x, and (ii) $|f_n(x)| \leqq |f_t(x, t_0+\theta n^{-1})| \leqq K$, by the mean value theorem and the boundedness of f_t. Apply Theorem 1.

THEOREM 5. *Let $f(x, y)$ be bounded, and suppose the iterated integrals $\int_c^d[\int_a^b f(x, y)dx]dy$ and $\int_a^b[\int_c^d f(x, y)dy]dx$ both exist. Then they are equal.*

Proof. By existence of the first integral, for instance, I mean that $f(x, y)$ is integrable in x for each y, and that the resulting function $F(y)=\int_a^b f(x, y)dx$ is integrable. Let $I_1, \cdots, I_r$ be a partition of $[a, b]$ and let $x_1 \in I_1, \cdots, x_r \in I_r$. Then $S(Y)=\sum_{i=1}^{r} f(x_i, y)|I_i|$ is an approximation to $F(y)$ for each y, while $\int_c^d S(y)dy=\sum_{i=1}^{r}\int_c^d f(x_i, y)dy|I_i|$ is an approximation to the second iterated integral in the theorem. Repeat this for a sequence of partitions with mesh max $|I_i|$ tending to 0 to form a sequence of functions $S_n(y)$ converging for each

y to $F(y)$, and apply Theorem 1. For hypothesis (ii) in Theorem 1 we have $|S_n(y)| \leqq K(b-a)$ where K is a bound for $|f(x, y)|$.

THEOREM 6. *Let* $\{f_n\}$ *be a sequence of functions defined on* $[a, \infty)$ *and integrable on* $[a, b]$ *for all* $b>a$. *Assume*

(i″) $f_n(x) \to f(x)$ *almost everywhere in* $[a, \infty)$, f *being integrable on every finite interval.*

(ii″) *There exists* k *defined on* $[a, \infty)$ *such that* $\int_a^\infty k$ *is convergent and* $|f_n| \leqq k$ *for all* n. *Then* $\int_a^\infty f_n \to \int_a^\infty f$.

Proof. These integrals are absolutely convergent by (ii″). Given $\epsilon>0$ find X so large that $\int_X^\infty k < \frac{1}{4}\epsilon$. Then apply Theorem 2 on the interval $[a, X]$ to find N so large that $\left|\int_a^X f_n - \int_a^X f\right| < \frac{1}{2}\epsilon$ when $n>N$. Then for these n we have

$$\left|\int_a^\infty f_n - \int_a^\infty f\right| \leqq \left|\int_a^X f_n - \int_a^X f\right| + \int_X^\infty |f_n| + \int_X^\infty |f| < \epsilon.$$

As a final example I shall use Theorem 6 to evaluate the conditionally convergent integral $\int_0^\infty x^{-1} \sin x dx$ which turns up in the theory of Fourier series and integrals. A formal trick which works is:

$$\begin{aligned}\int_0^\infty \frac{\sin x}{x} dx &= \int_0^\infty \sin x \int_0^\infty e^{-xt} dt dx \\ &= \int_0^\infty \left[\int_0^\infty \sin x e^{-xt} dx\right] dt \\ &= \int_0^\infty (1+t^2)^{-1} dt = \tfrac{1}{2}\pi.\end{aligned}$$

The change in the order of integration can be explained as follows. Let

$$\begin{aligned}f_n(t) &= \int_0^n \sin x e^{-xt} dx \\ &= (1+t^2)^{-1}[1 - (t \sin n + \cos n)e^{-nt}].\end{aligned}$$

Evidently $|f_n(t)| \leqq (1+t^2)^{-1}[1+(t+1)e^{-t}] = k(t)$, and $\int_0^\infty k$ converges. Thus Theorem 6 gives

$$\begin{aligned}\int_0^\infty \int_0^\infty \sin x e^{-xt} dx dt &= \lim \int_0^\infty f_n \\ &= \int_0^\infty (1+t^2)^{-1} dt\end{aligned}$$

as required. Note that $f_n(0) = 1 - \cos n$ does not converge, so that the "almost everywhere" version of hypothesis (i) is used. The convergence of $f_n(t)$ to $(1+t^2)^{-1}$ is not uniform even on $(0, \infty)$.

Reference

1. F. Hausdorff, Beweis eines Satzes von Arzelà, Mathematische Z., 26 (1927) 135–137.