Clearly, $a=x_0$ so that $b=f(x_0)$, and therefore $f(x_{n_j})\to f(x_0)$, which contradicts the assumed property of $\{x_n\}$.

[*Query*. Can you remove the hypothesis that A and B are *metric* spaces?]

An immediate trivial corollary of this is then the following basic result.

COROLLARY. *Let A and B be metric spaces, with A compact. Let f: $A\to B$ be continuous and one-to-one. Then f^{-1} is continuous.*

Proof. Let $B_0=f(A)$, a compact set in B. Then, f^{-1}: $B_0\to A$ is a function whose graph is homeomorphic to the graph of f. Since f is continuous, this set is compact, and f^{-1} is continuous.

At this point, one can point out the importance of the requirement that A be compact. Let $\mathcal{C}$ be the metrizable space consisting of all continuous real valued functions on $[0, 1]$ with the uniform convergence topology. Define a function F: $\mathcal{C}\to\mathcal{C}$ by $F(\phi)=\psi$ where $\psi(x)=\int_0^x\phi$, $0\leqq x\leqq 1$. It is easily checked that F is continuous. (Indefinite integration of a uniformly convergent sequence leaves it uniformly convergent, if we normalize at one point.) It is also one-to-one, for if $F(\phi_1)=F(\phi_2)$ then $\int_0^x(\phi_1-\phi_2)=0$ for all $x\in[0, 1]$, and $\phi_1=\phi_2$. However, F^{-1}—which exists as a function—is *not* continuous! (One sees that F^{-1} is D, the differentiation operator, and if $\psi_n\to\psi_0$ uniformly, it does not follow that ψ_n' converges to ψ_0'.)

Here, then, is a convenient example of a function F which is continuous, and whose graph is therefore closed, but whose graph is not compact. Likewise, D is a function whose graph is closed, but which is not continuous on $\mathcal{C}$. (A simpler example is the function g on $[0, 1]$ defined by $g(x)=1/x$, $g(0)=1$.).

An invited address at the MAA meeting in Denver, 1965.

References

1. W. Rudin, Principles of Mathematical Analysis, McGraw-Hill, New York, 1964, pp. 246–247.
2. A. Wilansky, Functional Analysis, Blaisdell, New York, 1964.

SIDE-AND-DIAGONAL NUMBERS

FREDERICK V. WAUGH and MARGARET W. MAXFIELD,
Arlington, Virginia and Gainesville, Florida

1. Purpose of this paper. Our purpose is not to present new methods. It is to discuss methods that were developed over 2,000 years ago. We hope that our presentation may be interesting, and that it may help some readers to understand the nature of irrational surds. But we do not claim any new discoveries. If there is anything new in this paper, it is in the presentation.

2. Side-and-diagonal numbers. The ancient Greeks had an ingenious way of approximating square roots. The Pythagoreans proved that the square root of 2 is "irrational,"—meaning that it is not the ratio between any two integers.

Euclid meant the same thing when he said that the diagonal of a square is "incommensurable with its side."

Since $\sqrt{2}$ is irrational, its exact value cannot be stated by any fraction. The best we can do in any number system is an approximation. Many of the early Greeks (including Aristarchus of Samos) used 7/5 as an approximation to $\sqrt{2}$. If the sides of a square were 5 inches, the diagonal would be $\sqrt{50}$ inches, or $\sqrt{(7^2+1)}$ inches,—or slightly more than 7 inches. That is what Plato [7, p. 309] meant when he said that 7 is "the rational diameter of 5", and that $\sqrt{50}$ is "the irrational diameter of 5".

But 7/5 is only a rough approximation to $\sqrt{2}$. Greek arithmeticians sought closer estimates. Since

$$(1) \qquad d/s \neq \sqrt{2}, \text{ i.e., } d^2/s^2 \neq 2, \text{ i.e., } d^2 \neq 2s^2,$$

they sought solutions of the equation

$$(2) \qquad d^2 - 2s^2 = \pm 1, \text{ i.e., } d^2 = 2s^2 \pm 1, \text{ i.e., } d^2/s^2 = 2 \pm 1/s^2.$$

This equation is a special case of "Pell's equation" $d^2 - ns^2 = N$. (See, for instance, [6] pp. 158–161.)

Clearly, the larger the numbers s and d satisfying (2), the closer d^2/s^2 is to 2, and the closer d/s is to $\sqrt{2}$. The Pythagoreans developed a systematic way of getting all the positive solutions of (2). This will be explained in the next section.

Probably the Greeks used similar methods to find close approximations to the square roots of many other numbers. When Archimedes [3, pp. 91–98] set out to estimate the value of π, he needed lower and upper bounds to $\sqrt{3}$. He simply stated, as Heath [3, p. lxxx] says, "without a word of explanation," that

$$(3) \qquad 265/153 < \sqrt{3} < 1351/780.$$

In the same work, Archimedes gave bounds to the square roots of seven large numbers—again without explanation. Perhaps Archimedes assumed that the Greek methods of bounding square roots was well known. And perhaps the Greeks used several methods. But, as we shall indicate in the next section of this paper, we know that one method was to form a sequence of fractions

$$(4) \qquad d_0/s_0,\ d_1/s_1,\ d_2/s_2,\ \cdots,\ d_t/s_t,$$

such that

a. each fraction could be computed from the one immediately preceding it, and

b. each successive fraction was a closer approximation to $\sqrt{n}$, —preferably alternating from greater than to less than $\sqrt{n}$.

The Greek mathematicians and philosophers were especially interested in $\sqrt{2}$, the ratio of the diagonal of a square to its side. Thus, in the sequence of fractions for $\sqrt{2}$, the numerator was known as the diagonal number and the denominator as the side number. In later years this general method of approximating the square root of any number became known as the "method of side-and-diagonal numbers," even though the geometric relations were different.

Günther [2] and Heath [3] have surveyed the literature concerning side-and-diagonal numbers. Günther's review is an exhaustive survey of all the literature on the subject up to 1882. Heath's treatment is shorter, but penetrating. Also, it is part of a most interesting discussion of Greek methods of arithmetic, in general. We shall here review the work of three men who have contributed greatly to the subject. Then we shall present a synthesis.

3. Theon of Smyrna. About 130 A.D., Theon of Smyrna wrote a book explaining Plato's mathematics. Theon of Smyrna is distinguished from Theon of Alexandria, who also was a mathematician, and who wrote about square roots. Dupuis [1, pp. 71–75], in 1892, translated Theon's book into French. That book is our best source of information about how the ancient Greeks used side-and-diagonal numbers to approximate the square root of 2.

Theon gave the following sequence of fractions

$$1/1,\ 3/2,\ 7/5,\ 17/12,\ \cdots \tag{5}$$

each term of which is an approximation to $\sqrt{2}$, and each term of which can be computed from the previous term by the recursion formula

$$d_{t+1}/s_{t+1} = \frac{2s_t + d_t}{s_t + d_t}\,. \tag{6}$$

One can easily continue the sequence. The next fraction would be $(2\cdot 12+17)/(12+17)=41/29$.

Theon discussed the sequence of fractions (5) and stated that $1^2=2\cdot 1^2-1$, $3^2=2\cdot 2^2+1$, $7^2=2\cdot 5^2-1$, and $17^2=2\cdot 12^2+1$. He concluded that, in general,

$$d_t^2 = 2\cdot s_t^2 \pm 1;\ 2 = (d_t^2 \mp 1)/s_t^2;\ \sqrt{2} = \sqrt{(d_t^2 \mp 1)}/s_t, \tag{7}$$

for any positive integer t. He also noted that the sign before the 1 alternated from $-$ to $+$, and he stated that the ratio d/s approached $\sqrt{2}$ as t increased.

Why the initial fraction 1/1? Theon said it is because unity is the principle of all figures and the generator of all numbers. That is Platonic metaphysics. Actually, one could start with any numbers d_0 and s_0 with $d_0^2-2s_0^2=e_0$. Then, using (7), he would get $d_t^2-2s_t^2=(-1)^te_0$. If d_0 and s_0 are integers, the smallest possible e_0 is ± 1. The modern mathematician would be satisfied with any initial fraction d/s, such that $d^2-2s^2=\pm 1$. He would accept 1/1, because $1^2-2\cdot 1^2=-1$.

Theon did not prove equation (7); he only showed that it held for the first several terms. Equation (7) can be proved by induction. From (6), $d_{t+1}^2=4s_t^2+4s_td_t+d_t^2$ and $2s_{t+1}^2=2s_t^2+4s_td_t+2d_t^2$ so that

$$d_{t+1}^2 - 2s_{t+1}^2 = 2s_t^2 - d_t^2 = -\ (d_t^2 - 2s_t^2). \tag{8}$$

Since, as Theon showed, the first several fractions in the sequence are such that $d_t^2-2s_t^2=\pm 1$, with the sign before the 1 alternating from $+$ to $-$, this will continue indefinitely, and since $d_t^2-2s_t^2=\pm 1$,

$$d_t^2/s_t^2 = 2 \pm 1/s_t^2, \tag{9}$$

for each positive integer t. As t increases, s_t increases, and its reciprocal approaches zero. Thus, d_t/s_t continuously approaches $\sqrt{2}$, and is alternately below and above the true value, as Theon said.

To anticipate further results, Theon's system can be written in terms of vectors and matrices. Let

$$v_0 = [s_0 \quad d_0] = [1 \quad 1] \quad \text{and} \quad T = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}. \tag{10}$$

Then

$$e_0 = 1^2 - 2\cdot 1 = |T| = -1. \tag{11}$$

The sequence of fractions is

$$v_1 = v_0T,\ v_2 = v_1T = v_0T^2, \cdots, v_{t+1} = v_tT = v_0T^t, \quad \text{and} \quad e_t = (-1)^{t+1}. \tag{12}$$

To get faster convergence, one could use

$$T^2 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}.$$

Here $|T^2| = 9-8 = +1$. If he started with $v_0 = [1 \ 1]$, he would get

$$v_1 = [5 \quad 7],\ v_2 = [29 \quad 41],\ v_3 = [169 \quad 239], \cdots.$$

In other words, the sequence would be 1/1, 7/5, 41/29, 239/169, $\cdots$, which approaches $\sqrt{2}$ monotonically from below. Similarly, if he started with $v_0 = [1\ 2]$, he would get a sequence approaching $\sqrt{2}$ monotonically from above.

A more rapidly converging sequence that oscillates about $\sqrt{2}$ can be had from

$$T^3 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^3 = \begin{bmatrix} 7 & 10 \\ 5 & 7 \end{bmatrix},$$

with $|T^3| = |T|^3 = (-1)^3 = -1$. Again, starting with $v_0 = [1\ \ 1]$, one would find

$$v_1 = [12 \quad 17],\ v_2 = [169 \quad 239], \cdots$$

from which one would know that $17/12 > \sqrt{2} > 239/169$.

Before leaving Theon of Smyrna, we note in passing that the sequence 1/1, 3/2, 7/5, 17/12, $\cdots$ can also be obtained from the continued fraction

$$\sqrt{2} = 1 + \frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2}\cdots, \tag{13}$$

since

$$1 = 1/1,\ 1 + \tfrac{1}{2} = \tfrac{3}{2},\ 1 + \frac{1}{2+}\,\frac{1}{2} = 7/5, \cdots.$$

We think that the ancient Greek method is far simpler than continued fractions.

4. de Lagny. M. de Lagny's paper [5] was written in 1723 and published in 1725. (Referees must have been slow even then!) It is still one of the most penetrating analyses of the uses of side-and-diagonal numbers.

Early in his paper (p. 57) de Lagny remarked that in approximating the square root of 2 "the best that any finite intelligence can do is to find *regularly, indefinitely, and without any fumbling* (notice these three conditions, and especially the last) is, I say, to find the sequence of all squares taken two by two, such that the difference between the larger and double the smaller be as small as possible, whether in excess or in default, and it is evident that this difference can not be less than unity."

Thus de Lagny says that the problem of approximating $\sqrt{2}$ is to find the sequence d_t/s_t, such that $d_t^2=2s_t^2\pm 1$, —and to find it by a regular process that can be continued indefinitely, and that does not require any *tâtonnement* (which we take to mean no fumbling around, no trial-and-error, no guessing).

He suggests that Archimedes might have found his bounds $265/153<\sqrt{3}<1351/780$ by using two sequences of fractions. The first sequence is

$$(14)\qquad 2/1,\ 7/4,\ 26/15,\ 97/56,\ 362/209,\ 1351/780,\ \cdots,$$

each term of which is greater than $\sqrt{3}$, and closer to $\sqrt{3}$ than the term immediately preceding it. The second sequence is

$$(15)\qquad 1/1,\ 5/3,\ 19/11,\ 71/41,\ 265/153,\ 980/571,\ \cdots,$$

each term of which is less than $\sqrt{3}$, and closer to $\sqrt{3}$ than the term immediately preceding it.

In either sequence (14) or (15) the terms after the first are

$$(16)\qquad \frac{d_{t+1}}{s_{t+1}}=\frac{3s_t+2d_t}{2s_t+d_t},$$

or

$$(17)\qquad v_{t+1}=\begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}v_t.$$

The sequence of fractions in (14) extended indefinitely by the recursion formula (16) gives all the positive solutions of $d^2-3s^2=+1$. The sequence in (15) gives all the positive solutions of $d^2-3s^2=-2$. To prove this, note that $d_{t+1}^2=9s_t^2+12s_td_t+4d_t^2$ and $3s_{t+1}^2=12s_t^2+12s_td_t+3d_t^2$ so that $d_{t+1}^2-3s_{t+1}^2=-3s_t^2+d_t^2$. This result is due to the fact that

$$\begin{vmatrix}2 & 3\\ 1 & 2\end{vmatrix}=+1.$$

Thus, if $e_t=d_t^2-3s_t^2$, $e_{t+1}=e_t=e$, and e stays constant as d_t and s_t increase. We see easily from the first term of (14) that $e=+1$. The first term in (15) shows that for each fraction in that sequence $e=-2$. It can be proved by the Greek method that there are not two integers s and d such that $d^2-3s^2=-1$.

The above development also proves convergence. Since the error term, e, is fixed for either sequence, we have $d_t^2-3s_t^2=e$, and $d_t^2/s_t^2=3+e/s_t^2$. But s_t increases with t, so d_t^2/s_t^2 approaches 3, and d_t/s_t approaches $\sqrt{3}$.

Several commentators have noted that it seems strange that Archimedes would have carried the first of de Lagny's sequences to 6 terms to get the very close bound $\sqrt{3}<1351/780$, but would have carried the second sequence to only 5 terms to get the much looser bound $\sqrt{3}>265/153$. They have searched for a single sequence in which 265/153 and 1351/780 would be successive terms. More on this later.

After discussing his two sequences of fractions approximating $\sqrt{3}$, de Lagny proceeded to discuss the general case of $\sqrt{n}$. He gave initial fractions and recursion formulas for approximating the square roots of 2, 3, 5, 6, 7, 8, 13, 41, and 43. He then described a general process for approximating $\sqrt{n}$. Let $a^2<n<b^2$, where a^2 and b^2 are fairly close approximations to n. (By trial squaring one can usually approximate to the nearest integer easily.) Let

$$s_0 = 1,\ d_0 = b,\ s_{t+1} = as_t + d_t,\ d_{t+1} = ns_t + ad_t. \tag{18}$$

This can be written

$$v_0 = [1 \quad b],\ T = \begin{bmatrix} a & n \\ 1 & a \end{bmatrix},\ |T| = a^2 - n < 0. \tag{19}$$

De Lagny noted that one can use either a or b for d_0 and either a or b for the diagonal entries of T, and so obtain different sequences. For a sequence with $|T|=a^2-n<0$, as in (19), d_t/s_t oscillates around $\sqrt{n}$ as it approaches $\sqrt{n}$. For a sequence with $|T|=b^2-n>0$, d_t/s_t increases monotonically toward $\sqrt{n}$ if $d_0/s_0=a/1<\sqrt{n}$, or decreases monotonically toward $\sqrt{n}$ if $d_0/s_0=b/1>\sqrt{n}$.

Note that for $t>0$, $d_{t+1}^2=n^2s_t^2+2ans_td_t+a^2d_t^2$ and $ns_{t+1}^2=na^2s_t^2+2ans_td_t+nd_t^2$ so that

$$d_{t+1}^2 - ns_{t+1}^2 = (a^2 - n)(d_t^2 - ns_t^2) = (a^2 - n)^t(b^2 - n). \tag{20}$$

Thus

$$d_t^2/s_t^2 = n + (a^2 - n)^{t-1}(b^2 - n)/s_t^2. \tag{21}$$

If the fraction at the right in (21) approaches zero as t increases, d_t/s_t approaches $\sqrt{n}$.

We note that de Lagny's sequence is useful for approximating the square roots of large numbers. Early in his "Measurement of a Circle" Archimedes [**3**, p. 94] stated that $\sqrt{(349450)}>591\ 1/8$. Try de Lagny's principle. The surd in question is less than 600 and greater than 590. Let $s_0=1$, $d_0=b=600$, or $v_0=[1\ 600]$, and

$$T = \begin{bmatrix} a & n \\ 1 & a \end{bmatrix} = \begin{bmatrix} 590 & 349450 \\ 1 & 590 \end{bmatrix}.$$

As the determinant of this T is negative, we see from (21) that d_t^2/s_t^2 oscillates around n. Therefore, since $\sqrt{(349450)}<d_0/s_0=600/1$, we have $\sqrt{(349450)}>d_1/s_1=703{,}450/1{,}190=591\ 16/119>591\ 1/8$.

All of Archimedes' bounds to square roots could have been obtained in similar fashion, using side-and-diagonal numbers in essentially the way de Lagny suggested.

5. Heilermann. Heilermann [4], in 1881, suggested that the square root of any number n could be approximated by a series of side-and-diagonal numbers starting with

$$d_0/s_0 = 1/1 \tag{22}$$

and using the recursion formula

$$\frac{d_{t+1}}{s_{t+1}} = \frac{ns_t + d_t}{s_t + d_t}. \tag{23}$$

In other words, Heilermann let

$$v_0 = [s_0 \quad d_0] = [1 \quad 1] \quad \text{and}$$
$$T = \begin{bmatrix} 1 & n \\ 1 & 1 \end{bmatrix}. \tag{24}$$

Using (19) and (20) with $a=1$, we have for Heilermann's fractions $d_t^2/s_t^2 = n + (1-n)^t/s_t^2$. Thus Heilermann's d_t/s_t approaches $\sqrt{n}$ as t increases if $(1-n)^t/s_t^2$ approaches zero.

Heilermann used this procedure to get a single sequence of fractions to bound $\sqrt{3}$. The sequence is

$$\begin{aligned} 1/1,\ 4/2 &= 2/1,\ 5/3,\ 14/8 = 7/4,\ 19/11,\ 52/30 = 26/15,\ 71/41,\ 194/112 \\ &= 97/56,\ 265/153,\ 724/418 = 362/209,\ 989/571,\ 2702/1560 \\ &= 1351/780, \cdots \end{aligned} \tag{25}$$

Note that whenever the numerator and denominator of a fraction contain a common factor, it is canceled before the next step. Thus, 4/2 is reduced to 2/1 before the next fraction is computed.

The even-numbered terms of Heilermann's sequence are identical to de Lagny's first sequence: 2/1, 7/4, 26/15, $\cdots$, while the odd-numbered terms are de Lagny's second sequence: 1/1, 5/3, 19/11, $\cdots$. The reason for this is that de Lagny's transformation matrix T_L is one-half the square of Heilermann's T_H. Thus

$$T_H = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad T_L = \frac{1}{2}\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}.$$

In this way, de Lagny's sequence skips every other term of Heilermann's.

Archimedes' bounds to $\sqrt{3}$ are the 9th and 12th terms of (25). Heilermann [4, pp. 122–123] recognized that Archimedes would probably not have used the rather loose bound 265/153 if he had computed the 12 terms to get the close bound 1351/780. Rather, he would have used 989/571. Therefore, Heilermann sought a modification of the method that would give Archimedes' bounds as two successive fractions in the same sequence.

To do this, he first found a sequence for $\sqrt{(27/25)} = 3\sqrt{3}/5$. Starting with

$$v_0 = [s_0 \quad d_0] = [1 \quad 1] \quad \text{and} \quad T = \begin{bmatrix} 1 & 27/25 \\ 1 & 1 \end{bmatrix},$$

he got the series 1/1, 26/25, 53/51, 1351/1300, $\cdots$. To approximate $\sqrt{3}$ any of these fractions would be multiplied by 5/3. Thus, the sequence for $\sqrt{3}$ would be

$$5/3,\ 26/25\cdot 5/3 = 26/15,\ 53/51\cdot 5/3 = 265/153,\ 1351/1300\cdot 5/3 = 1351/780,\ \cdots .$$

Eureka! Archimedes' two bounds are terms 3 and 4 of the same series. Heath [**3**, p. 97] wrote, "This is one of the very few instances of success in bringing out the two Archimedean approximations in immediate sequence without any foreign values intervening. No other methods appear to connect the two values in this direct way except those of Hunrath and Hultsch depending on the formula

$$a \pm b/2a > \sqrt{(a^2 \pm b)} > a \pm b/(2a \pm 1)."$$

Without wishing to detract in any way from the importance of Heilermann's excellent development, we note that the transformation

$$\frac{1}{2} T_H^3 = \frac{1}{2}\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}^3 = \begin{bmatrix} 5 & 9 \\ 3 & 5 \end{bmatrix}$$

provides a rapidly convergent sequence with desirable properties. Because its determinant is -2, the sequence oscillates about $\sqrt{3}$, as does the sequence generated by T_H; yet with the use of $T_H^3/2$ we get only every third term of Heilermann's sequence, so that convergence is rapid, and incidentally, Archimedes' bounds are adjacent in the sequence.

Starting with $v_0 = [3\ 5]$, we get directly the sequence for $\sqrt{3}$:

$$5/3,\ 52/30 = 26/15,\ 265/152,\ 2702/1560 = 1351/780,\ \cdots .$$

In general,

$$T_H^2 = \begin{bmatrix} 1 & n \\ 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1+n & 2n \\ 2 & 1+n \end{bmatrix}, \quad \text{and}$$

$$T_H^3 = \begin{bmatrix} 1 & n \\ 1 & 1 \end{bmatrix}^3 = \begin{bmatrix} 1+3n & n(3+n) \\ 3+n & 1+3n \end{bmatrix}.$$

Whenever n is odd, each element of T_H^2 and of T_H^3 is divisible by 2, so we can use $T_H^2/2$ and $T_H^3/2$ as transformation matrices. This reduces the size of our fractions, and also reduces the error term. In fact, when d_t and s_t are divided by k, the error term is divided by k^2. For example, the first term of the above sequence, 5/3, gives an error $e_0 = 5^2 - 3\cdot 3^2 = -2$. If we use 52/30 as the second term we get $e_1 = 52^2 - 3\cdot 30^2 = +4$. But, if we use the equivalent fraction 26/15, we get $e_1 = 26^2 - 3\cdot 15^2 = +1$. If we use the reduced fractions, the errors are successively $-2, +1, -2, +1, \cdots$. If none of the fractions had been reduced, the errors would have been $-2, +4, -8, +16, \cdots$. Note, however, that the *relative* error, $e_t/s_t^2 = (d_t/s_t)^2 - n$, is unchanged by removal of the common factor.

6. Synthesis. The methods of Theon, de Lagny, and Heilermann are all of one generic kind. In general, let d_0/s_0 be an approximation to $\sqrt{n}$. The sequence of side-and-diagonal numbers is computed from $v_1 = Tv_0$, $v_2 = Tv_1 = T^2v_0, \cdots$, where $v_0 = [s_0 \; d_0]$ and

$$T = \begin{bmatrix} d_0 & ns_0 \\ s_0 & d_0 \end{bmatrix}. \tag{26}$$

Thus, the sequence of fractions is

$$\frac{d_0}{s_0}, \; \frac{d_1}{s_1} = \frac{ns_0^2 + d_0^2}{2s_0d_0}, \; \frac{d_2}{s_2} = \frac{ns_0s_1 + d_0d_1}{d_0s_1 + s_0d_1}, \; \cdots. \tag{27}$$

Using methods explained above, one can easily show that $d_{t+1}^2 - ns_{t+1}^2 = (d_0^2 - ns_0^2)(d_t^2 - ns_t^2) = (d_0^2 - ns_0^2)^t$. The system converges to $\sqrt{n}$ if $(d_0^2 - ns_0^2)^t/s_t^2$ approaches zero as t increases.

We can also get this sequence s_t, d_t by exponentiating $d_0 - s_0\sqrt{n}$, with $d_t - s_t\sqrt{n} = (d_0 - s_0\sqrt{n})^{t+1}$. If we choose s_0, d_0 so that $d_0 - s_0\sqrt{n} = k$, with $-1 < k < +1$, we have $d_t - s_t\sqrt{n} = k^{t+1}$, which approaches zero as t increases, because $|k| < 1$. Then certainly $d_t/s_t - \sqrt{n} = k^{t+1}/s_t$ approaches zero as t increases, since s_t increases, proving that d_t/s_t approaches $\sqrt{n}$.

From $d_{t+1} - s_{t+1}\sqrt{n} = (d_t - s_t\sqrt{n})(d_0 - s_0\sqrt{n})$, we have $d_{t+1} = d_0d_t + ns_0s_t$ and $s_{t+1} = s_0d_t + d_0s_t$, so that the exponentiation process accomplishes the same transformation as (26):

$$T = \begin{bmatrix} d_0 & ns_0 \\ s_0 & d_0 \end{bmatrix}.$$

Computing Archimedes' bounds to $\sqrt{3}$ by the system

$$v_0 = [3 \quad 5] \quad \text{and} \quad T = \begin{bmatrix} 5 & 9 \\ 3 & 5 \end{bmatrix}$$

can, then, be accomplished equivalently by exponentiating $5 - 3\sqrt{3}$:

$$(5 - 3\sqrt{3})^2 = 26 - 15\sqrt{3}, \; (5 - 3\sqrt{3})^3 = 265 - 153\sqrt{3},$$

$$(5 - 3\sqrt{3})^4 = 1351 - 780\sqrt{3}.$$

We know that $-1 < 5 - 3\sqrt{3} < +1$, so the sequence 5/3, 26/15, 265/153, 1351/780, $\cdots$ converges. We already knew this, of course, from a different standpoint.

References

1. J. Dupuis, Théon de Smyrne: exposition des connaissances mathématiques utiles pour la lecture de Platon, Hachette, Paris, 1892.

2. Siegmund Günther, Die Quadratischen Irrationalitäten der Alten, Abhandlungen zur Geschichte der Mathematik, Leipzig, 1882, pp. 1–134.

3. Sir Thomas Heath, The Works of Archimedes, Dover, New York, undated.

4. Heilermann, Bemerkungen zu den Archimedischen Näherungswerthen der irrationalen Quadratwurzeln, Zeitschrift für Mathematik und Physik, Leipzig, 1881, pp. 121–126.

5. M. de Lagny. Méthode Générale pour transformer les Nombres irrationnaux en séries de Fractions rationnelles les plus simples et les plus approchantes qu'il soit possible, Histoire de l'Académie Royale des Sciences, Paris, 1725, pp. 55–69.

6. Ivan Niven and H. S. Zuckerman, Introduction to the Theory of Numbers, Wiley, New York, 1960, pp. 158–161.

7. Plato, The Works of Plato, Jowett translation, Tudor, New York, undated.

ON FINITE RINGS

ROLANDO E. PEINADO, State University of Iowa and
University of Puerto Rico-Mayaguez

This note determines all finite rings whose additive group is a cyclic group. This result enables us to determine all finite rings, with a fixed additive group, having n elements, where n is a square-free integer and to determine when such rings are radical rings and when they are semisimple rings. We shall use $|S|$ to denote the cardinality of a finite set S. $\tau(n)$ represents the number of positive integral divisors of n. $\phi(n)$ is Euler's phi-function, that is, the number of positive integers less than n and relatively prime to n. Z indicates the ring of rational integers. Z_n is the ring of rational integers modulo n. (p, q) means the greatest common divisor of the integers p and q. Isomorphism means a one-to-one onto homomorphism.

It is clear that when the additive group of a finite ring R is a cyclic group generated by a nonzero element a, that is to say $\{a\}=R$, then the multiplicative structure of the ring is determined by a positive integer p such that $a^2=pa$. Henceforth, R_p represents a finite ring with $|R_p|=n$, whose additive group is cyclic and where $a^2=pa$. Let $\mathcal{R}_p$ be the isomorphism class formed by all finite rings isomorphic to R_p. It is clear that if $\mathcal{R}$ is the class of all finite rings whose additive group is a cyclic group of order n, $\mathcal{R}$ is the disjoint union of the $\mathcal{R}_p$.

Now let R_q belong to $\mathcal{R}_p$. Then there exists an isomorphism $\alpha: R_q \to R_p$, given by $a\alpha = xa$, x in Z. By properties of congruence relations on Z_n, it is easy to show that $(x, n)=1$. This implies that the members of $\mathcal{R}_p$ are given by considering all solutions y to the equation $px \equiv y \pmod{n}$, with $(x, n)=1$. Hence, if $d_1=(q, n)$ and $d_2=(p, n)$, then $d_1=qs+nt$ and $d_2=pr+nh$, for s, t, r, and h in Z; and since $px\equiv q \pmod{n}$, we have $q=px+nu$ for u in Z. Thus $d_1=pxs+nv$ and, therefore, $d_2|d_1$. But if R_q belongs to $\mathcal{R}_p$, then R_p belongs to $\mathcal{R}_q$. Therefore, $qz\equiv p \pmod{n}$ and $p=qz+nw$ for w in Z. Hence, $d_2=qzr+ng$ for g in Z. Thus $d_1|d_2$ and we have that $d_1=d_2$. Conversely, if $(p, n)=(q, n)=d$, then $q=q_1d$ and $d=pr+nt$ for r and t in Z. Thus $q=dq_1=prq_1+ntq_1$ and the congruence $px\equiv q$ $\pmod{n}$ has d incongruent solutions with at least one of them relatively prime to n. Therefore R_q belongs to $\mathcal{R}_p$. We have proved

THEOREM 1. *R_q belongs to $\mathcal{R}_p$ if and only if $(p, n)=(q, n)$.*