A PROOF OF MINKOWSKI'S INEQUALITY FOR CONVEX CURVES

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1. Introduction. There is a beautiful proof of the isoperimetric inequality in the plane by L. A. Santaló [7, pp. 38–39]. This is based on the following ideas. Given an oval in the plane, move a circle of suitable radius, counting for each position how many times it intersects the oval. The average number of intersections can be computed in two ways, and the result is the isoperimetric inequality with an explicit error term.

This proof of Santaló has been reproduced several times and is fairly well known. What is not so widely known is that the proof may be modified to yield a proof of the Minkowski inequality on mixed areas, and actually yields an improvement of this inequality which is due to Bonnesen. See Blaschke [2, pp. 33–36].

The purpose of this paper is to give an exposition of the theory of closed convex plane curves, mixed area, and the Minkowski inequality. The prerequisites are a slight acquaintance with convex bodies in the plane and the beginnings of the differential geometry of plane curve theory now included in most vector oriented calculus books. In the next section we review the elementary differential geometry of closed convex curves.

In the last section we state the form taken by the Minkowski inequality in three-space. Alas, no analogue of the Santaló proof is known and each of the known proofs is much harder than the one we give shortly for the plane case. Those wishing to explore the subject further will find plenty in the list of the references at the end which includes more than the works cited here.

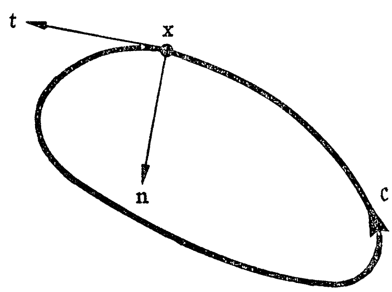


FIG. 1

2. Preliminaries. Let $\mathfrak{c}$ be a smooth closed convex curve in E^2 with positive curvature. Let s be the arc length, $\mathbf{x}=\mathbf{x}(s)$ the moving point in $\mathfrak{c}$, $\boldsymbol{t}=\boldsymbol{t}(s)$ the moving unit tangent, and $\boldsymbol{n}=\boldsymbol{n}(s)$ the moving unit normal (Fig. 1). The Frenet formulas are

$$\frac{d\mathbf{x}}{ds} = \boldsymbol{t}, \qquad \frac{d\boldsymbol{t}}{ds} = \kappa\boldsymbol{n}, \qquad \frac{d\boldsymbol{n}}{ds} = -\kappa\boldsymbol{t}. \tag{2.1}$$

Here $\kappa=\kappa(s)$ is the curvature which we are assuming satisfies $\kappa>0$ everywhere. If the total length of the curve is L, then $\mathbf{x}$ is a periodic vector function of s with fundamental period L and the other functions $\boldsymbol{t}$, $\boldsymbol{n}$, κ also have period L, although not necessarily as a fundamental period. Note that $\boldsymbol{n}$ is the inward drawn normal so that $\boldsymbol{t}$, $\boldsymbol{n}$ is a right-handed frame.

Of course we have

$$L=\oint_{\mathfrak{c}} ds. \tag{2.2}$$

There is also a line integral for the area A enclosed by $\mathfrak{c}$. If we write $d\mathbf{x}=(dx, dy)=\boldsymbol{t}ds$, then a rotation by angle $\pi/2$ leads to

$$\begin{aligned}(-dy, dx) &= \boldsymbol{n}ds,\\ (x, y)\cdot(-dy, dx) &= \mathbf{x}\cdot\boldsymbol{n}ds,\\ y\,dx - x\,dy &= -\mathbf{x}\cdot\boldsymbol{n}\,ds.\end{aligned}$$

Because of the well-known relation

$$A=\frac{1}{2}\oint_{\mathfrak{c}}(ydx-xdy),$$

an immediate consequence of Green's Theorem, we have

$$A=-\frac{1}{2}\oint_{\mathfrak{c}}\mathbf{x}\cdot\boldsymbol{n}ds. \tag{2.3}$$

This formula is valid for any simply closed curve and has nothing whatever to do with the convexity. We exploit the fact that $\mathfrak{c}$ is convex by introducing the parameter θ, the angle the outward drawn normal $-\boldsymbol{n}$ makes with the fixed x-axis. It is convenient to take the origin inside $\mathfrak{c}$. Since the curve $\mathfrak{c}$ is turning continuously ($d\boldsymbol{t}/ds=\kappa\boldsymbol{n}$, $\kappa>0$), each point $\mathbf{x}$ of the curve has a unique θ (modulo 2π) associated with it and θ makes a complete circuit, $0\leqq\theta\leqq 2\pi$ as $0\leqq s\leqq L$. (Fig. 2). The *support function* p is the distance of the tangent line at $\mathbf{x}$ from 0.

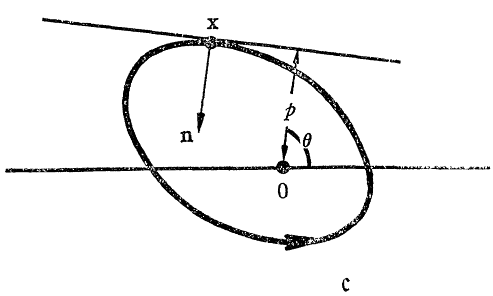


FIG. 2

We write $p=p(\theta)$ and

$$p = -\mathrm{x}\cdot n. \tag{2.4}$$

The function $p=p(\theta)$ has period 2π.

Analytically (this means as usual that, picturesque drawings to the contrary, all vectors start at 0) we have

$$\begin{aligned} t &= (-\sin\theta, \cos\theta),\\ n &= (-\cos\theta, -\sin\theta). \end{aligned} \tag{2.5}$$

The second formula is a direct consequence of the definition of θ and the first follows from this by rotating $\pi/2$.

After this we shall always denote $d/d\theta$ by ($'$). Clearly

$$t' = n, \qquad n' = -t. \tag{2.6}$$

Thus $dt=nd\theta$. But $dt=\kappa nds$ by (2.1). Hence

$$d\theta = \kappa ds. \tag{2.7}$$

Of course this is the standard interpretation of the curvature at the rate of turning of the tangent. Next we differentiate (2.4): $p'=-\mathrm{x}'\cdot n+\mathrm{x}\cdot t$. Since x' is parallel to t ($\mathrm{x}'=s'd\mathrm{x}/ds=s't$), $\mathrm{x}'\cdot n=0$ and so

$$\mathrm{x}\cdot t = p'. \tag{2.8}$$

Differentiating again, $\mathrm{x}'\cdot t+\mathrm{x}\cdot n=p''$. But $\mathrm{x}\cdot n=-p$ and $\mathrm{x}'\cdot t=(s't)\cdot t=s'$, hence

$$s' = p + p''. \tag{2.9}$$

But (2.7), $s'=1/\kappa=\rho$, the radius of curvature so this formula may be rewritten

$$\rho = p + p''. \tag{2.9$'$}$$

We now transform the integrals (2.2), (2.3):

$$L = \oint ds = \int_0^{2\pi} \rho d\theta = \int_0^{2\pi} (p + p'')d\theta.$$

Since $p''d\theta=d(p')$ is exact and p' has period 2π, $\oint p''d\theta=0$ and we have $L=\int_0^{2\pi} pd\theta$. For the area we have

$$A = -\frac{1}{2}\oint \mathrm{x}\cdot nds = \frac{1}{2}\int_0^{2\pi} p(p + p'')d\theta.$$

Since $d(pp')=pp''d\theta+p'^2d\theta$ we have $\int_0^{2\pi} pp''d\theta=-\int_0^{2\pi} p'^2d\theta$. We have obtained

$$L = \int_0^{2\pi} pd\theta, \qquad A = \frac{1}{2}\int_0^{2\pi} (p^2 - (p')^2)d\theta. \tag{2.10}$$

It is worth noting that p determines $\mathfrak{c}$ completely. In fact, by (2.4), (2.8), and (2.5),

$$x = p'\boldsymbol{t} - p\boldsymbol{n} = (-p' \sin \theta + p \cos \theta, p' \cos \theta + p \sin \theta).$$

Geometrically, the tangents to $\mathfrak{c}$ envelop $\mathfrak{c}$. To say it another way, the convex set K whose boundary is $\mathfrak{c}$ is the intersection of the half-planes including K determined by the lines of support (tangents). This means

$$K = \{v \in E^2 \mid -v\cdot\boldsymbol{n}(\theta) \leqq p(\theta) \text{ for all } \theta\}. \tag{2.11}$$

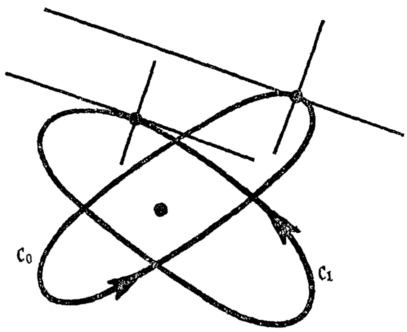


FIG. 3

MIXED AREAS. For this concept we work with two closed convex curves $\mathfrak{c}_0$ and $\mathfrak{c}_1$. Thus we have two vector functions $x_0 = x_0(\theta)$, $x_1 = x_1(\theta)$ of θ. By writing them this way we automatically set up a one-one correspondence between the curves whereby corresponding points have the same normal $\boldsymbol{n} = \boldsymbol{n}(\theta)$. (See Fig. 3.) The support functions are

$$p_0(\theta) = -x_0\cdot\boldsymbol{n}, \qquad p_1(\theta) = -x_1\cdot\boldsymbol{n}.$$

We now study a new convex curve obtained by translating $\mathfrak{c}_1$ to all possible positions such that the origin of $\mathfrak{c}_1$ is on $\mathfrak{c}_0$. To understand this (Fig. 4) we think of $\mathfrak{c}_1$ as the boundary of a rigid lamina which we are free to slide by translations only over the plane in which $\mathfrak{c}_0$ is fixed. We let the origin of $\mathfrak{c}_1$ in this lamina slide along $\mathfrak{c}_0$. Then $\mathfrak{c}_1$ envelops a curve (two curves actually; we take the outside one as illustrated). It is clear that the moving point of contact of this new curve $\mathfrak{c}$ with a particular translate of $\mathfrak{c}_1$ has the same direction as its moving origin has on $\mathfrak{c}_0$. We immediately conclude that our new curve $\mathfrak{c}$ has support function

$$p = p_0 + p_1.$$

Another way to look at this is through the Minkowski sum of convex sets. Let $\mathfrak{c}_0 = \partial K_0$, $\mathfrak{c}_1 = \partial K_1$ so that K_0, K_1 are convex regions bounded by $\mathfrak{c}_0$ and $\mathfrak{c}_1$ respectively. The Minkowski sum is the set K defined by

$$K = K_0 + K_1 = \{v_0 + v_1 \mid v_0 \in K_0, v_1 \in K_1\}. \tag{2.12}$$

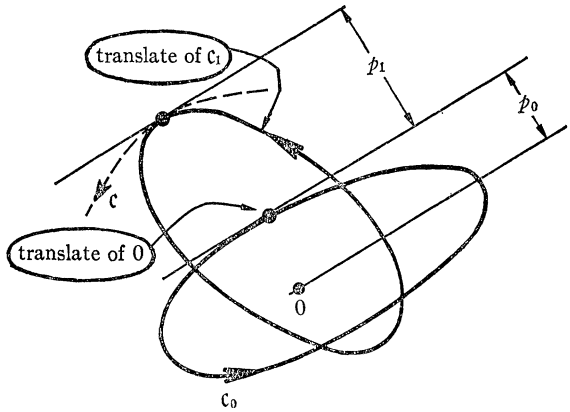


FIG. 4

By (2.11) applied to both K_0 and K_1, if $v_0 \in K_0$ and $v_1 \in K_1$, then

$$-(v_0 + v_1)\cdot \boldsymbol{n}(\theta) = -v_0\cdot\boldsymbol{n}(\theta) - v_1\cdot\boldsymbol{n}(\theta) \leqq p_0(\theta) + p_1(\theta).$$

But $\mathrm{x}_0(\theta) \in K_0$, $\mathrm{x}_1(\theta) \in K_1$, hence $\mathrm{x}_0(\theta) + \mathrm{x}_1(\theta) \in K$ and

$$-[\mathrm{x}_0(\theta) + \mathrm{x}_1(\theta)]\cdot\boldsymbol{n}(\theta) = p_0(\theta) + p_1(\theta).$$

This shows that K is the intersection of all the half-planes

$$\{v \mid -v\cdot\boldsymbol{n}(\theta) \leqq p_0(\theta) + p_1(\theta)\}$$

so that K is convex and its boundary $\mathfrak{c} = \partial K$ has support function $p = p_0 + p_1$.

(2.13) LEMMA. *The length and area of* $\mathfrak{c}$ *are given by*

$$L = L_0 + L_1, \qquad A = A_0 + 2A_{01} + A_1,$$

where L_i, A_i *are the length and area of* $\mathfrak{c}_i$ $(i=0, 1)$ *and* $A_{01} = \frac{1}{2}\int_0^{2\pi}[p_0p_1 - p_0'p_1']d\theta$ *is the* mixed area *of* $\mathfrak{c}_0$ *and* $\mathfrak{c}_1$.

Proof.

$$L = \int_0^{2\pi} p\,d\theta = \int_0^{2\pi}(p_0 + p_1)d\theta = \int_0^{2\pi} p_0\,d\theta + \int_0^{2\pi} p_1\,d\theta = L_0 + L_1.$$

$$A = \tfrac{1}{2}\int_0^{2\pi}(p^2 - p'^2)d\theta$$

$$= \tfrac{1}{2}\int_0^{2\pi}[(p_0^2 + 2p_0p_1 + p_1^2) - (p_0'^2 + 2p_0'p_1' + p_1'^2)]d\theta$$

$$= \tfrac{1}{2}\left\{\int_0^{2\pi}(p_0^2 - p_0'^2)d\theta + 2\int_0^{2\pi}(p_0p_1 - p_0'p_1')d\theta + \int_0^{2\pi}(p_1^2 - p_1'^2)d\theta\right\}$$

$$= A_0 + 2A_{01} + A_1.$$

It is clear that everything in sight is symmetric. If we slide the origin of $\mathfrak{c}_0$ along $\mathfrak{c}_1$ we get the same curve $\mathfrak{c}$ simply because $p_1+p_0=p_0+p_1$. Also $A_{10}=A_{01}$. We may obtain an unsymmetric formula for A_{01} by integrating the exact differential

$$d(p_0p_1') = (p_0'p_1' + p_0p_1'')d\theta.$$

This yields

$$\oint p_0'p_1'\,d\theta = -\oint p_0p_1''\,d\theta.$$

Thus

$$A_{01} = \frac{1}{2}\int_0^{2\pi} p_0(p_1 + p_1'')d\theta. \tag{2.14}$$

By (2.9) and (2.9′) we have

$$A_{01} = \frac{1}{2}\int_0^{2\pi} p_0\rho_1 d\theta = \frac{1}{2}\oint p_0 ds_1, \tag{2.15}$$

and similarly

$$A_{01} = \frac{1}{2}\int_0^{2\pi} p_1\rho_0 d\theta = \frac{1}{2}\oint p_1 ds_0. \tag{2.16}$$

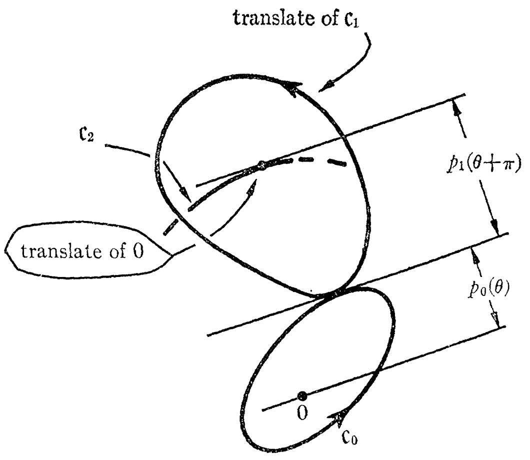


FIG. 5

Instead of translating $\mathfrak{c}_1$ so that the origin moves along $\mathfrak{c}_0$, let us try the following. We translate $\mathfrak{c}_1$ so that it is in contact with $\mathfrak{c}_0$ externally. Then the locus of the translated origin traces a new curve $\mathfrak{c}_2$ (Fig. 5). From the figure, this is a convex curve with support function p_2 given by

$$p_2(\theta) = p_0(\theta) + p_1(\theta + \pi).$$

Now $p_1(\theta+\pi)$ is the support function of the curve $\mathfrak{c}_1^*$ obtained by reflecting $\mathfrak{c}_1$ in the origin or, equivalently, by rotating $\mathfrak{c}_1$ through angle π. Thus the curve $\mathfrak{c}_2$ is obtained simply by applying our previous construction to $\mathfrak{c}_0$ and $\mathfrak{c}_1^*$ and (2.13) applies.

(2.17) LEMMA. *The length and area of* $\mathfrak{c}_2$ *are given by*

$$L_2 = L_0 + L_1, \qquad A_2 = A_0 + 2\overset{*}{A}_{01} + A_1$$

where

$$\begin{aligned} \overset{*}{A}_{01} &= \tfrac{1}{2}\int_0^{2\pi} [p_0(\theta)p_1(\theta + \pi) - p_0'(\theta)p_1'(\theta + \pi)]d\theta \\ &= \tfrac{1}{2}\oint p_1(\theta + \pi)ds_0(\theta). \end{aligned}$$

Proof. It is clear both geometrically and analytically that $\mathfrak{c}_1^*$ and $\mathfrak{c}_1$ have the same length and same area so the formula is a consequence of (2.13). The last expression comes from (2.16) applied to $p_1^*(\theta) = p_1(\theta+\pi)$.

3. The Minkowski inequality. The main result we are after is the Minkowski inequality which is the two-dimensional version of the Brunn-Minkowski inequality. We refer to Bonnesen-Fenchel [**3**, Sect. 49, 51] and Hadwiger [**6**, Chapt. 4] for other treatments of this and more general results.

(3.1) MINKOWSKI INEQUALITY. *If* $\mathfrak{c}_0$ *and* $\mathfrak{c}_1$ *are closed convex curves with areas* A_0 *and* A_1 *respectively and mixed area* A_{01}, *then* $A_{01}^2 \geqq A_0A_1$.

Note that in case $\mathfrak{c}_1$ is the unit circle we have $A_1 = \pi$, $p_1 = 1$, and by (2.16)

$$A_{01} = \tfrac{1}{2}\oint ds_0 = \tfrac{1}{2}L_0,$$

so the Minkowski inequality specializes to

$$\tfrac{1}{4}L_0^2 \geqq \pi A_0 \tag{3.2}$$

which is the classical isoperimetric inequality.

We also can state precisely when there is equality in (3.1).

(3.3) SUPPLEMENT. *If* $A_{01}^2 = A_0A_1$ *then* $\mathfrak{c}_0$ *and* $\mathfrak{c}_1$ *are homothetic, i.e., they differ by a dilatation and translation.*

Our proof will yield the following stronger form of (3.1) which makes (3.3) obvious. To state it we need the ideas of relative inradius and relative circumradius. The curves $\mathfrak{c}_0$ and $\mathfrak{c}_1$ are given and we write as before $\mathfrak{c}_0 = \partial K_0$, $\mathfrak{c}_1 = \partial K_1$.

The *inradius of* $\mathfrak{c}_0$ *relative to* $\mathfrak{c}_1$ is the largest real number r_0 such that a translate of r_0K_1 is in K_0 (Fig. 6). The *circumradius of* $\mathfrak{c}_0$ *relative to* $\mathfrak{c}_1$ *is the* smallest real number R_0 such that a translate of R_0K_1 contains K_0. Obviously $R_0 \geqq r_0$ with equality if and only if K_0 is a translate of r_0K_1 or, to say it another way, K_0 and K_1 are similar and similarly placed. Note that if $\mathfrak{c}_1$ is the unit circle then r_0 and R_0 are the ordinary inradius and circumradius of $\mathfrak{c}_0$.

(3.4) THEOREM. *Let* $\mathfrak{c}_0$ *and* $\mathfrak{c}_1$ *be closed convex curves with areas* A_0 *and* A_1 *respectively and mixed area* A_{01}. *Let* r_0, *resp.* R_0, *be the inradius, resp. circumradius, of* $\mathfrak{c}_0$ *relative to* $\mathfrak{c}_1$. *Then*

$$A_{01}^2 - A_0A_1 \geqq \frac{A_1^2}{4}(R_0 - r_0)^2.$$

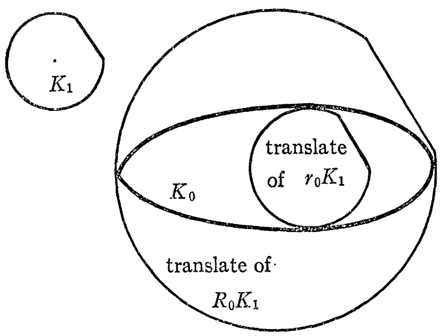
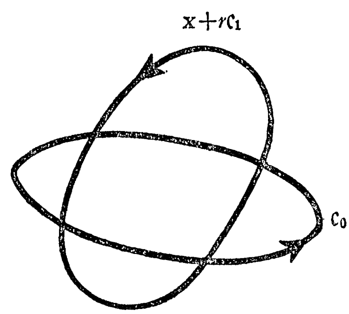


FIG. 6



Example: $N(\mathfrak{x})=4$.

FIG. 7

4. The proof. We fix a number r with $r_0 \leqq r \leqq R_0$. For each point $\mathfrak{x}=(x, y)$ of the plane we consider the translate $\mathfrak{x}+r\mathfrak{c}_1$ of $r\mathfrak{c}_1$. We are interested in the number $N(\mathfrak{x})$ of points of intersection of this translate with $\mathfrak{c}_0$. (See Fig. 7.) Thus

$$N(\mathfrak{x}) = \left| (\mathfrak{x} + r\mathfrak{c}_1) \cap \mathfrak{c}_0 \right|. \tag{4.1}$$

We shall average this. Precisely, we set

$$I = \iint_{E^2} N(\mathfrak{x})dxdy. \tag{4.2}$$

The function $N(\mathfrak{x})$ may be infinite for certain values of $\mathfrak{x}$ such that the curves $\mathfrak{c}_0$ and $\mathfrak{x}+r\mathfrak{c}_1$ have a point of tangency. Now the points of $\mathfrak{x}$ for which there is such a point of tangency lie on two curves which constitute a set of zero area. Consequently it makes no difference what $N(\mathfrak{x})$ is equal to on these curves, the integral I is insensitive to these values.

The outside curve (Fig. 8) is precisely the curve $\mathfrak{c}_2$ we studied in Section 2 (see Fig. 5) but with $\mathfrak{c}_1$ replaced by $r\mathfrak{c}_1$. If x lies outside of this curve then $N(\mathbf{x})=0$. But if x lies inside of this curve, $N(\mathbf{x})>0$. For otherwise either $x+r\mathfrak{c}_1$ entirely surrounds $\mathfrak{c}_0$, a contradiction to $r\leqq R_0$, or $\mathfrak{c}_0$ entirely surrounds $x+r\mathfrak{c}_1$, a contradiction to $r_0\leqq r$.

Actually if we ignore the boundary $\mathfrak{c}_2$ and the other curve ($\mathfrak{c}_0$ and $\mathbf{x}+r\mathfrak{c}_1$ tangent internally) where $N(\mathbf{x})$ may have nasty values we may assert that $N(\mathbf{x})\geqq 2$.

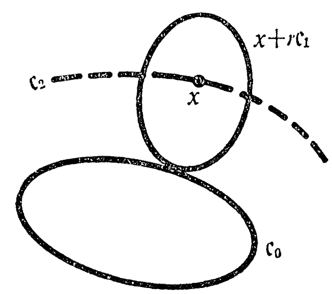


FIG. 8

For when two closed curves intersect at all (and really cross, no common tangents) then they intersect an even number of times. What comes in must go out.

According to (2.17) applied to $\mathfrak{c}_0$ and $r\mathfrak{c}_1$ we have for the area of K_2 where $\mathfrak{c}_2=\partial K_2$,

$$A_2 = A_0 + 2rA_{01}^{*} + r^2A_1.$$

From these remarks we have

$$I = \int\int_{E^2} N(\mathbf{x})dxdy = \int\int_{K_2} N(\mathbf{x})dxdy \geqq \int\int_{K_2} 2dxdy \geqq 2A_2,$$

$$I \geqq 2(A_0 + 2rA_{01}^{*} + r^2A_1). \tag{4.3}$$

One should worry why the discrete valued function $N(x)$ has an integral. It is easiest to ignore it, but a word to the wise in the ways of measure theory will be sufficient: the sets where $n(\mathbf{x})=2m$ are open and E^2 is the join of these and a couple of piecewise smooth curves.

The relation (4.3) gives a lower bound for I. We now go after a precise evaluation of I. This is done in three steps. Step 1: we find all translates of $r\mathfrak{c}_1$ passing through a fixed point. Step 2: we find all translates of $r\mathfrak{c}_1$ which intersect a fixed short segment. Step 3: we break up $\mathfrak{c}_0$ into many short segments (or use an inscribed polygon approximation) and sum. Now the details!

If $\mathbf{x}_0$ is a fixed point, the moving origin of $r\mathfrak{c}_1$ traces the oval $\mathbf{x}_0-r\mathfrak{c}_1$ as $r\mathfrak{c}_1$ is translated to all positions such that it passes through $\mathbf{x}_0$, i.e., as $\mathbf{x}_1(\theta)$ is translated

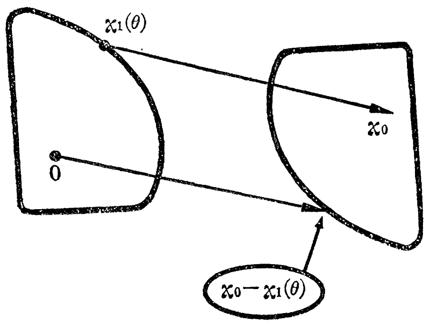


FIG. 9

to $\mathbf{x}_0$, the origin gets translated to $\mathbf{x}_0-\mathbf{x}_1(\theta)$. See Fig. 9.

Now consider a short segment of length ds_0. Think of this as a part of the fixed curve $\mathfrak{c}_0$. Which translates of $r\mathfrak{c}_1$ intersect this segment? Those for which the center of the translate lies in the thin strip swept out by $\mathbf{x}_0-r\mathfrak{c}_1$ as $\mathbf{x}_0$ moves its distance ds_0 along the segment (Fig. 10). Ignoring the small shaded area at

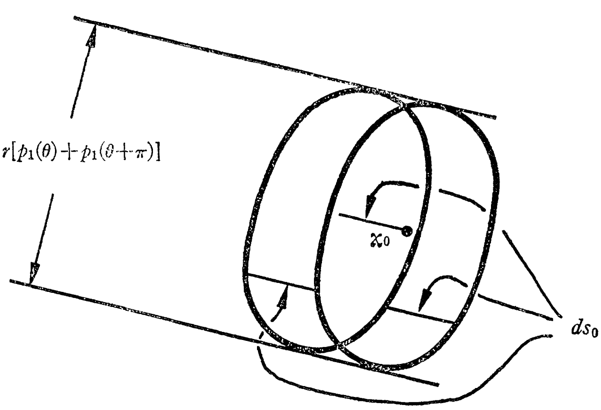


FIG. 10

the end, which is of smaller order of magnitude, the area of each half of this strip is obtained by multiplying its base $r[p_1(\theta)+p_1(\theta+\pi)]$ by its constant height ds_0. Now except for the shaded areas, each point of this strip is the center of a translate of $r\mathfrak{c}_1$ which intersects the segment once. Thus the contribution of this segment to the integral I is

$$2r[p_1(\theta) + p_1(\theta + \pi)]ds_0 + O(ds_0)^2.$$

Summing and passing to the limit we have

$$I = 2r\int_0^{2\pi}[p_1(\theta) + p_1(\theta + \pi)]ds_0. \tag{4.4}$$

By (2.16) and (2.17) we may write this as

$$I = 4r(A_{01} + \overset{*}{A}_{01}), \tag{4.5}$$

which completes our exact evaluation of I.

By (4.3) and (4.5) we have

$$2rA_{01} \geqq A_0 + r^2A_1 \tag{4.6}$$

for all r such that $r_0 \leqq r \leqq R_0$.

Now the existence of a single real number r such that the polynomial $x^2A_1 - 2xA_{01} + A_0$ is less than or equal to zero at $x = r$ implies this polynomial has real roots and hence positive discriminant $A_{01}^2 - A_0A_1 \geqq 0$. This is the Minkowski inequality which is good enough. However, we learn more by exploiting the fact that the polynomial is nonpositive on the whole interval $r_0 \leqq r \leqq R_0$. Simplest is to complete the square to rewrite (4.6) as

$$A_{01}^2 - A_0A_1 \geqq (rA_1 - A_0)^2.$$

We substitute for r the two extreme values and slyly change the sign inside the square:

$$A_{01}^2 - A_0A_1 \geqq (R_0A_1 - A_{01})^2,$$

$$A_{01}^2 - A_0A_1 \geqq (A_{01} - r_0A_1)^2.$$

We average these inequalities and use the elementary fact that the average of the squares is greater than or equal to the square of the average:

$$A_{01}^2 - A_0A_1 \geqq \left\{\tfrac{1}{2}[(R_0A_1 - A_{01}) + (A_{01} - r_0A_1)]\right\}^2,$$

so finally $A_{01}^2 - A_0A_1 \geqq A_1^2(R_0 - r_0)^2/4$.

5. Remarks.

1. If $\mathfrak{c}_1$ is a circle of radius r then $L_1 = 2\pi r$, $A_1 = \pi r^2$, $A_{01} = \frac{1}{2}rL_0$. The curve $\mathfrak{c}$ of (2.13) is the oval parallel to $\mathfrak{c}_0$ at distance r and (2.13) yields Steiner's formulas

$$L = L_0 + 2\pi r, \qquad A = A_0 + rL_0 + \pi r^2.$$

2. If $\mathfrak{c}_1$ is the unit circle, then (3.4) specializes to

$$L_0^2 - 4\pi A_0 \geqq \pi^2(R_0 - r_0)^2$$

which is the inequality of Bonnesen mentioned in the Introduction. This is a striking relation between the length, area, circumradius and inradius of an oval.

3. Of course (3.4) implies $A_{01}^2 \geqq A_0 A_1$. It also implies that if $A_{01}^2 = A_0 A_1$, then $R_0 = r_0$ and (3.3) follows.

4. The estimate (4.3) may be replaced by the more precise

$$I = 2(A_0 + 2r\overset{*}{A}_{01} + r^2 A_1) + 2 \sum_{m=2}^{\infty} (m-1) f_m,$$

where f_m is the area of the set on which $N(x) = 2m$. This follows from

$$I = \sum_1^{\infty} 2m f_m \quad \text{and} \quad \sum_1^{\infty} f_m = A_2.$$

It may be used to improve (4.6) and the subsequent inequalities.

6. The situation in space. Let K be a compact convex body in E^3 whose boundary $\mathfrak{S} = \partial K$ is a smooth closed convex surface with positive total curvature at each point. (This means that if n is the outward unit normal at x, then $x \rightarrow n$ is a one-one smooth mapping of $\mathfrak{S}$ onto the unit sphere whose inverse map is also smooth.) In addition to the volume V of K and the surface area A of $\mathfrak{S}$ there is another invariant,

$$M = \iint_S H dA, \tag{6.1}$$

where H is the mean curvature. The isoperimetric inequality in E^3, due to H. A. Schwarz, is

$$A^3 \geqq 36\pi V^2. \tag{6.2}$$

This may be sharpened to

$$A^3 - 36\pi V^2 \geqq [\sqrt{A} - \sqrt{4\pi}\, r_0]^6, \tag{6.3}$$

where r_0 is the inradius. This is enough to show that equality holds in (6.2) only for a sphere. Another refinement of (6.2) consists of the pair of inequalities

$$A^2 \geqq 3MV, \tag{6.4}$$

$$M^2 \geqq 4\pi A. \tag{6.5}$$

These imply both (6.2) and the further result

$$M^3 \geqq 48\pi^2 V. \tag{6.6}$$

These and (6.3) may all be summarized in the assertion that the function

$$f(t) = (V + At + Mt^2 + \tfrac{4}{3}\pi t^3)^{1/3} \tag{6.7}$$

is concave ($f'' \leqq 0$).

Now let K_0, K_1 be two convex bodies of the type considered. The first remarkable fact is that the volume of a linear combination $\lambda K_0 + \mu K_1 (\lambda, \mu \geqq 0)$ is given by a polynomial

(6.8) $$|\lambda K_0 + \mu K_1| = V_0\lambda^3 + 3V_{001}\lambda^2\mu + 3V_{011}\lambda\mu^2 + V_1\mu^3.$$

This defines the *mixed* volumes V_{001}, V_{011}. The function $f(t)^3$ from (6.7) is the special case $K_0 = K$, K_1 = sphere of radius t. The Brunn-Minkowski inequality is the assertion that

(6.9) $$|K_0 + tK_1|^{1/3}$$

is a concave function of t, $0 \leqq t$. This has the consequences

(6.10) $$V_{001}^2 \geqq V_0 V_{011},$$

(6.11) $$V_{011}^2 \geqq V_{001} V_1,$$

which generalize (6.4) and (6.5) and furthermore

(6.12) $$V_{001}^3 \geqq V_0^2 V_1,$$

(6.13) $$V_{011}^3 \geqq V_0 V_1^2$$

which generalize (6.4) and (6.5).

This provides a glimpse into a large and fascinating part of geometry.

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