NONSTANDARD SET THEORY

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Infinitely small and infinitely large quantities were systematically introduced into mathematics with the invention of calculus by Newton and Leibniz. The use of such quantities, however, was accompanied by logical contradictions, which mathematicians of the seventeenth and eighteenth centuries were unable to resolve. Although the method of infinitesimals generally yielded correct results, no one ever succeeded in formulating a precise, noncontradictory set of rules governing these objects; and infinitesimal quantities were gradually displaced (at least, in pure mathematics) by the familiar ε-δ calculus. A mathematically sound model of infinitely small and infinitely large objects became possible only after advances in mathematical logic in the twentieth century. Nonstandard Analysis, developed by A. Robinson in 1960, not only provided foundations for the calculus of infinitesimals in the classical spirit but also enabled mathematicians to use "nonstandard" objects in ways that could not be attempted on the basis of vague, intuitive understanding alone. Since then, interesting applications were found in various branches of mathematics, mathematical physics, and economics.

Robinson's exposition in [**10**] and its subsequent simplifications unfortunately involve the cumbersome apparatus of mathematical logic. Our aim here is to present methods of Nonstandard Analysis at a level of formalism customary in other branches of mathematics. We view nonstandard objects as ideal, imaginary elements adjoined to the universe of the standard mathematics and formulate a few simple and reasonably intuitive principles governing their behavior. We then show, on examples selected to illustrate a variety of nonstandard constructions, how nonstandard mathematics can be developed from these principles.

The basic framework for Nonstandard Analysis is presented in §§1–3; this system was introduced in [**3**], where its relative consistency with respect to the Zermelo-Fraenkel set theory is shown. We examine the real line and some concepts of general topology from our point of view in §§4–5; these results can be found in Robinson [**10**] and Luxemburg [**7**], [**8**]. Section 6 is devoted to nonstandard measure theory; our approach is basically that of Loeb [**6**] (except that we construct Loeb's extension in Theorem 3 of §6 directly, rather than using Carathéodory's Theorem), with some ideas coming from Anderson [**1**]. The final §7, part I, contains a more formal description of the logical foundations, and parts II and III discuss the relationship between our approach and the classical one based on higher-order nonstandard models, as well as some other axiomatizations of Nonstandard Analysis.

1. The standard universe. Our starting point is the universe of objects ordinarily studied by "standard" mathematicians—numbers, sets, geometric figures—together with the usual relations between such objects: order, set membership, etc. In order to distinguish these objects from other entities by which we enrich the mathematical universe, we qualify them by using the adjective *standard*. For example, 0, 1 and 17 are standard natural numbers, $\sqrt{2}$ and π are standard real numbers, R is the standard set of all real numbers, $\in$ and $<$ are standard relations, etc. Notation $\mathcal{S}(A)$ is sometimes used to express the fact that A is a standard object. A property or a statement is called *standard* if all quantified variables in it range over standard objects. Standard concepts (relations, operations, constants) are the ones defined by standard properties. They can then be used in other standard statements. (Here, as well as in the rest of the paper, we do not distinguish between properties and their descriptions in some formal

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language. So, "properties" are the same things as "formulas." Statements are properties (formulas) with no parameters (free variables). The reader desiring a more exact explanation of these matters might at this point begin to read part I of §7, and keep referring to it in §§2 and 3.)

We illustrate these ideas on some typical examples.

1. The relation of inclusion, $\subseteq$, is a standard concept defined for all standard sets A and B by the standard property,

For every standard x, $x \in A$ implies $x \in B$.

2. Continuity of a standard function $f: R \to R$ at a standard point $a \in R$ is a standard property of f and a; namely:

> For every standard $\varepsilon > 0$, $\varepsilon \in R$, there is a standard $\delta > 0$, $\delta \in R$, such that, for all standard $x \in R$, $|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$.

Indeed, we see that all quantified variables range over standard objects, and all concepts mentioned by this property (such as $>$, $\in$, 0, R, $|\cdot|$, $-$) are standard; either primitive ($\in$) or else understood to be defined, in their turn, by their usual standard properties.

We conclude with an observation that all theorems of standard mathematics are (presumably true) standard statements in our sense.

2. The internal universe. We now adopt a point of view that, besides the standard, "real" mathematical objects, there are also nonstandard, "ideal" objects, possessing formally the same properties as the standard ones. For a picturesque example, consider the set B of all birds; B is a standard set having eagles, storks, and sparrows among its standard elements. From the nonstandard standpoint, however, B has also fictitious elements, such as phoenixes. The nonstandard elements of B have all the properties of the standard ones: they are bipeds, females lay eggs, etc.

In concurrence with the established practice, the objects that are either standard or nonstandard are called *internal*. Notation $\mathcal{I}(A)$ means that A is an object from the universe of internal sets; so, of course, $\mathcal{S}(A)$ implies $\mathcal{I}(A)$, but not vice versa. Our first principle is intended to make precise the idea that nonstandard objects have formally the same properties as the standard ones.

We say that a property $\Phi^{\mathcal{I}}$ is an *internalization* of the standard property Φ if $\Phi^{\mathcal{I}}$ is obtained from Φ by replacing all quantified variables ranging (as they have to) over standard objects by variables ranging over internal objects.

PRINCIPLE OF EMBEDDING. *Standard objects $A, B, \ldots$ have a standard property Φ if and only if they have the internalized property $\Phi^{\mathcal{I}}$.*

To clarify this principle and its use, let us consider a few examples.

1. The standard relation of inclusion is defined by the statement:

For all standard A and B, $A \subseteq B$ if and only if for all standard x, $x \in A$ implies $x \in B$. (1)

By the Principle of Embedding, this statement is equivalent to its internalization:

For all internal A and B, $A \subseteq B$ if and only if for all internal x, $x \in A$ implies $x \in B$. (2)

We see from (2) that the standard relation $\subseteq$, originally defined only for pairs of standard sets, becomes, from the nonstandard viewpoint, applicable to all pairs of internal sets, and has the intended meaning. If, in particular, A and B are standard sets, then it follows from (1) and (2) that all standard elements of A belong to B if and only if all internal elements of A belong to B. In other words, the meaning of $A \subseteq B$ for standard A, B is the same, whether we look at A and B from the standard or from the nonstandard point of view; and we could use either (1) or (2) as a definition of $\subseteq$.

2. Let N be the standard set of natural numbers; we define N by the following property:

$$\begin{aligned}&\text{(i) } 0=\varnothing\in N;\\&\text{(ii) for all standard } n, \text{ if } n\in N, \text{ then } n+1=n\cup\{n\}\in N;\\&\text{(iii) if } N' \text{ is any standard set having properties (i) and (ii), then } N\subseteq N'.\end{aligned}\tag{3}$$

The internalization of (3) then asserts:

$$\begin{aligned}&\text{(i) } 0=\varnothing\in N;\\&\text{(ii) for all internal } n, \text{ if } n\in N, \text{ then } n+1=n\cup\{n\}\in N;\\&\text{(iii) if } N' \text{ is any internal set having properties (i) and (ii), then } N\subseteq N'.\end{aligned}\tag{4}$$

Briefly, the standard mathematician defines N as the smallest standard set containing zero and, with each standard element, also its successor. The Principle of Embedding shows that this same set N can also be defined internally as the smallest internal set containing zero and, with each internal element, also its successor. While (3) is a basis for proofs by standard induction, (4) can be used to justify proofs by internal induction (see the beginning of §3). Notice also that the operation of successor is well defined for all internal sets; again due to the Principle of Embedding: Since for every standard set n there is a unique standard set $n+1=n\cup\{n\}$, we can conclude that for every internal set n there is a unique internal set $n+1=n\cup\{n\}$. We also see that the standard arithmetic properties of natural numbers carry over into the internal universe. For example,

$$\text{For all standard natural numbers } n \text{ and } m,\ n+m=m+n.\tag{5}$$

is a standard theorem; therefore

$$\text{for all internal natural numbers } n \text{ and } m,\ n+m=m+n.\tag{6}$$

is an (internal) theorem. It is in this sense that the internal natural numbers (and internal objects in general) have the same properties as the corresponding standard ones.

3. Let us consider the standard definition of continuity from §1, Example 2. Continuity is a standard concept; by the Principle of Embedding, it is automatically applicable to internal functions and internal points:

> An internal function $f: R\to R$ is continuous at an internal point $a\in R$ if and only if for every internal $\varepsilon>0$, $\varepsilon\in R$, there is an internal $\delta>0$, $\delta\in R$, such that, for all internal $x\in R$, $|x-a|<\delta$ implies $|f(x)-f(a)|<\varepsilon$.

Moreover, for standard f and a the internal definition is equivalent to the standard one. Here, of course, the standard operations of absolute value and subtraction, the standard relations $\in$ and $<$, and other standard concepts, are applicable to internal elements, as in analogous instances in Examples 1 and 2. Similarly, the expression "an internal function f" is merely a shorthand for "f is some internal object which is a function," where "being a function" is a standard property, whose internalized definition reads: "An internal set of ordered pairs f is a function if, for all internal a, b_1, b_2, $(a,b_1)\in f$ and $(a,b_2)\in f$ imply $b_1=b_2$."

4. Let $\Phi(x,y)$ be a standard property. For any standard A and y, $\{\text{standard } x\in A|\Phi(x,y)\}$ denotes the standard set of all standard elements of A with the property Φ; that is, for all standard z,

$$z\in\{\text{standard } x\in A|\Phi(x,y)\} \text{ if and only if } z\in A \text{ and } \Phi(z,y).$$

The Principle of Embedding makes the standard operation $\{\text{standard } x\in A|\Phi(x,y)\}$ applicable to all internal A and y, with the intended result: for all internal z,

$$z\in\{\text{standard } x\in A|\Phi(x,y)\} \text{ if and only if } z\in A \text{ and } \Phi^{\mathfrak{I}}(z,y).$$

From now on, we always employ the more suggestive notation $\{\text{internal } x \in A \mid \Phi^{\mathcal{G}}(x,y)\}$ for this set, whenever either A or y is nonstandard.

These examples show how the Principle of Embedding automatically extends the scope of each standard concept into the internal universe, in such a way that the property which defines the concept over the standard universe (i.e., with variables ranging over the standard objects) also defines it over the internal universe (i.e., with variables ranging over the internal objects after internalization). There is also a useful reformulation of the Principle of Embedding from a somewhat different angle:

Let Φ be a standard property. If there is an internal object having the property $\Phi^{\mathcal{G}}$, then there is a standard object having the property Φ.

Proof is simple. "There is an internal x such that $\Phi^{\mathcal{G}}(x)$" is an internal statement, and is therefore equivalent to the standard statement of which it is an internalization: "There is a standard x such that $\Phi(x)$."■

For example, if we prove that a standard function f is continuous at some internal point z, we can conclude that f is continuous also at some standard point x.

Next we have to address the question of *existence* of nonstandard objects; it is easy to see that the Principle of Embedding alone does not imply existence of any. Our professed aim is to enrich the standard universe by all possible sorts of imaginary elements. As an example, we would like to add infinitesimals, that is, nonstandard real numbers ι such that $\iota \neq 0$, but $|\iota| < 1/n$ for every standard $n \in N$. This is a typical way of describing ideal elements; one formulates a list of properties, say, $\Phi(0,\iota)$, $\Phi(1,\iota)$, $\Phi(2,\iota), \ldots$, and requires the ideal element to satisfy all of them simultaneously. (In our example, $\Phi(0,\iota)$ is "$\iota \in R$ and $\iota \neq 0$" and $\Phi(n,\iota)$ is "$|\iota| < 1/n$," for standard $n \neq 0$.) Obviously, this will not always work; if we assume existence of ι satisfying simultaneously $\Phi(0,\iota)$ and $\Phi(1,\iota)$ where $\Phi(0,\iota)$ is "$\iota \in N$" and $\Phi(1,\iota)$ is "$\iota \notin N$", we get a contradiction. We can postulate existence of an object satisfying simultaneously a given list of properties only if these properties are mutually consistent. Guided by the principle that whatever can be consistently imagined exists (as an idea, though not necessarily as a "real," standard object), and, more pragmatically, by a desire to have as many ideal elements available as possible, we formulate

THE WEAK PRINCIPLE OF SATURATION. *Let Φ be a standard property. If A is a standard set, and if for every standard finite $a \subseteq A$ there is a standard y such that $\Phi(x,y)$ holds simultaneously for all standard $x \in a$, then there is an internal y such that $\Phi^{\mathcal{G}}(x,y)$ holds simultaneously for all standard $x \in A$.*

As an example, we prove existence of infinitesimals. Let $\Phi(x,y)$ be the property "($x=0$ and $y \in R$ and $y \neq 0$) or ($x \neq 0$ and $|y| < 1/x$)," and let $A = N$ be the standard set of natural numbers. Then for every standard finite $a \subseteq N$ there is a standard y, namely, $1/(\max(a)+1)$, which satisfies Φ simultaneously for all standard $x \in a$. If y is the internal object provided by the Principle of Saturation, then $\Phi^{\mathcal{G}}(x,y)$ holds for all standard $x \in N$, that is, $y \in R$, $y \neq 0$ and $|y| < 1/x$ for all standard $x \in N (x \neq 0)$. This is precisely what we require of infinitesimals.

The intuitive rationale for our formulation of the Principle of Saturation goes along the lines of the previous discussion. If we assume that, for any standard finite $a \subseteq A$, there is a standard y such that $\Phi(x,y)$ holds for all standard $x \in a$, then this y satisfies $\Phi^{\mathcal{G}}(x,y)$ simultaneously for all standard $x \in a$($\Phi(x,y)$ holds if and only if $\Phi^{\mathcal{G}}(x,y)$ holds, by the Principle of Embedding), and each finite subcollection of the properties $\Phi(x,y)$, $x \in A$, x standard, is consistent. Thus, intuitively, the whole collection is consistent (any proof of a contradiction can use only finitely many of the properties).

We now further illustrate the use of the Principle of Saturation by proving two simple theorems of Nonstandard Analysis.

THEOREM 1. *Every standard infinite set has nonstandard elements.*

Proof. Let A be standard and infinite. For every $a \subseteq A$, standard and finite, there is a standard $y \in A \setminus a$. Then "$y \in A$ and $y \neq x$" holds simultaneously for all standard $x \in a$. We conclude, on the basis of the Principle of Saturation, that there is an internal y such that "$y \in A$ and $y \neq x$" holds simultaneously for all standard $x \in A$. But this means that y is a nonstandard element of A.■

THEOREM 2. *For every standard set A there is an internal finite set $\alpha \subseteq A$ containing all standard elements of A.*

Proof. If $a \subseteq A$ is standard and finite, then there is a standard set y such that "$y \subseteq A$, y is finite, and $x \in y$" holds simultaneously for all standard $x \in a$ (let $y = a$). By the Principle of Saturation, there is an internal set α such that $\alpha \subseteq A$, α is finite, and $x \in \alpha$ holds for all standard $x \in A$.■

Theorem 2 may appear at first almost contradictory, as it seemingly asserts that a finite set can contain an infinite set. Let us examine it in some detail. Since α is an internal finite set, it has, from the point of view of an internal mathematician, all properties of finite sets. In particular, every internal subset of α is finite. Therefore, if A is infinite, then A is not a subset of α. All standard elements of A belong to α, but there must be nonstandard elements of A which do not belong to α. Also, α itself is then nonstandard (because standard sets having the same standard elements are equal). So far so good, but let us look now at the set ${}^\circ A$, whose elements are precisely the standard elements of A: ${}^\circ A = \{x \in A \mid \mathcal{S}(x)\}$. Obviously ${}^\circ A \subseteq \alpha$ and ${}^\circ A$ is infinite (if A is). The threat of a contradiction is resolved by realizing that existence of ${}^\circ A$ does not follow from the principles we accepted so far. (Notice that the property used to define ${}^\circ A$ is not internal—see §7, part I, for details.) In other words, *${}^\circ A$ is not an internal set*! The above argument simply shows existence of entities outside of the internal universe. It should not be surprising that, for some *external* point of view, the nonstandard finite sets contain infinite subsets and, indeed, are themselves infinite.

Noninternal entities are useful for better understanding of internal objects, as a notational convenience, and as an essential tool in some constructions. We therefore devote the next paragraph to the principles of work with them.

3. The external universe. We begin with an example. The standard set of natural numbers N contains the standard elements $0, 1, 2, \ldots$, but, besides them, also some nonstandard elements (see Theorem 1 in §2). Let us now consider a collection ${}^\circ N$, whose elements are precisely the standard natural numbers; clearly ${}^\circ N \subseteq N$, but ${}^\circ N \neq N$. We see immediately that ${}^\circ N$ is not standard (if two standard sets have the same standard elements, they are equal). Moreover, ${}^\circ N$ is not even internal. This can be proved by internal induction, as presented in (4) of Example 2 in §2. It is clear that $0 \in {}^\circ N$. If an internal $n \in {}^\circ N$, then n is standard, and therefore $n+1 \in {}^\circ N$. If ${}^\circ N$ were internal, we would have $N \subseteq {}^\circ N$, a contradiction.

Another proof of this result may provide additional insight. Since $<$ is a standard, therefore internal, well-ordering of N, every nonempty internal subset of N has a $<$-least element. If ${}^\circ N$ were internal, $N \setminus {}^\circ N$ would be a nonempty internal subset of N; but $N \setminus {}^\circ N$ is the collection of all nonstandard natural numbers, and does not have a least element (if $n \in N \setminus {}^\circ N$, then $(n-1) \in N \setminus {}^\circ N$).

It is useful to be able to work with objects such as ${}^\circ N$. We therefore extend our universe of discourse further by adding to it also some noninternal objects. We call objects which are either internal or noninternal, *external* (here we differ from Robinson [**10**], who uses "external" in the sense of our "noninternal"). One should visualize the external objects as subsets of the universe of internal objects (of course, some of them are already internal), sets of such subsets, sets of sets of such subsets, etc. (a cumulative process well known to set theorists). Thus all noninternal objects are sets, and the scope of the set membership relation naturally extends to them. Internal objects may be members of external sets, but noninternal objects cannot belong to internal sets

(the internal universe was formed prior to its extension by noninternal objects). The familiar intuitive arguments, used to motivate the axioms of the standard set theory, can be used here, too, to justify validity of

THE PRINCIPLES OF ZERMELO SET THEORY WITH THE AXIOM OF CHOICE

in the external universe. We do not state these axioms here (see [3]); for practical purposes, this means merely that all elementary properties of sets used in topology, measure theory, functional analysis, etc., are valid in our universe of external sets. As we point out in [3, Theorem 3], Fraenkel's Axiom Schema of Replacement is false in our universe, and one has to avoid constructions which depend on it. The only such constructions of possible interest to nonlogicians occur in the theory of ordinal and cardinal numbers, and extra care is needed there (for example, not every external well-ordering is isomorphic to an external ordinal). However, we know of no use for external infinite ordinals and cardinals in Nonstandard Analysis; since we can assume that all of the axioms of Zermelo-Fraenkel set theory with Choice are valid in the standard universe, there are no obstacles to the study of standard and internal ordinals and cardinals in the usual way.

An important question is the relationship between internal concepts and the corresponding external ones. It is no longer true, as in the case of standard vs. internal, that each internal concept automatically agrees with its external counterpart. Nevertheless, with little effort one can prove that it is so for most of the elementary set-theoretic relations and operations, and we give a few examples below. We use the usual notation for the internal concepts, and a superscript $\mathcal{E}$ for the corresponding concepts defined in the external universe.

1. For any external sets A and B, $A \subseteq^{\mathcal{E}} B$ means that for all external x, $x \in A$ implies $x \in B$. If now A and B are internal, then all elements of A and B are also internal, and $A \subseteq^{\mathcal{E}} B$ is equivalent to $A \subseteq B$ (i.e., to the statement: for all internal x, $x \in A$ implies $x \in B$). The relation of inclusion defined in the external universe therefore agrees, on internal sets, with the previously defined relation of inclusion in the internal universe, and we can (and do) drop the superscript $\mathcal{E}$ and write simply $A \subseteq B$, whether A and B are internal or not.

2. The same sort of argument shows that, for example,

$$A \,{}^{\mathcal{E}}\!\cup B = A \cup B \qquad A \,{}^{\mathcal{E}}\!\cap B = A \cap B$$

$${}^{\mathcal{E}}\{A,B\} = \{A,B\} \qquad A \,{}^{\mathcal{E}}\!\times B = A \times B$$

f is a $\mathcal{E}$-function if and only if f is a function

$<$ is a $\mathcal{E}$-ordering if and only if $<$ is an ordering; ${}^{\mathcal{E}}\bigcup_{\iota \in I} A_\iota = \bigcup_{\iota \in I} A_\iota$, etc.,

whenever A, B, f, $<$ and $\langle A_\iota | \iota \in I \rangle$ are internal sets. In all these and similar cases, we drop the unnecessary superscript $\mathcal{E}$.

3. An example of a concept for which the two definitions diverge is the operation of power-set. If we define

$$X \in {}^{\mathcal{E}}\mathcal{P}(A) \text{ if and only if } X \subseteq A \text{ and}$$

$$X \in \mathcal{P}(A) \text{ if and only if } X \text{ is internal and } X \subseteq A,$$

then obviously $\mathcal{P}(A) \subseteq^{\mathcal{E}} \mathcal{P}(A)$; however, ${}^{\circ}N \in {}^{\mathcal{E}}\mathcal{P}(N) \setminus \mathcal{P}(N)$, and so the two concepts do not agree even on standard sets.

4. If Φ is any property (not necessarily standard, or even internal) and A, y are sets, $\{x \in A | \Phi(x,y)\} = {}^{\mathcal{E}}\{x \in A | \Phi(x,y)\}$ denotes the external set of all external $x \in A$ with the property Φ. We have to distinguish carefully between this notion and that of $\{\text{standard } x \in A | \Phi(x,y)\}$

defined (so far) only for standard Φ, A and y. For example, if $\Phi(x)$ is the standard property "for some standard n, $x \leqslant n$," $\{\text{standard } x \in N | \Phi(x)\} = \{\text{standard } x \in N | x \leqslant n \text{ for some standard } n\} = N$, while $\{x \in N | \Phi(x)\} = {}^{\mathcal{E}}\{x \in N | x \leqslant n \text{ for some standard } n\} = {}^\circ N$. The point here is that {standard ...} is being evaluated in the standard (or, equivalently, internal) universe, where we do not separate the standard elements from the nonstandard ones, while $\{\ldots\}$ is evaluated in the external universe, where we do.

We further clarify and extend conventions dealing with the set abstraction symbol later in this paragraph. First, however, we need to introduce one other important principle (our last):

THE PRINCIPLE OF STANDARDIZATION. *For every external set A there is a standard set $^*A$ such that A and $^*A$ have the same standard elements.*

We first note that use of the notation $^*A$ is justified, because the set in question is uniquely determined by A: If $^{**}A$ is another standard set having the same standard elements as A, then $^*A$ and $^{**}A$ have the same standard elements, and so $^*A = {}^{**}A$.

The Principle of Standardization does not assert anything about the nonstandard elements of A; they may or may not belong to $^*A$, and $^*A$ may have nonstandard elements which do not belong to A. (However, no noninternal elements of A may belong to $^*A$, since $^*A$ is internal.) Obviously, $^*A = A$ if A is standard; on the other hand, $^*({}^\circ N) = N \neq {}^\circ N$. For any A, we define $^\circ A$ to be the external *set of all standard elements of* A; that is, $^\circ A = \{x \in A | x \text{ is standard}\}$. If A is standard, we have $^\circ A \subseteq A$ and $^*({}^\circ A) = A$.

Returning now to our discussion of the abstraction symbol, consider a set A and a property Φ. Then $B = \{x \in A | \Phi(x)\}$ is an external set, and $^*B$ is a standard set with the property that a *standard* x belongs to $^*B$ if and only if $x \in A$ and $\Phi(x)$. We denote $^*B$ by $\{\text{standard } x \in A | \Phi(x)\}$, rather than $^*\{x \in A | \Phi(x)\}$, and notice that for standard Φ and A, this agrees with our previous conventions.

In summary: Given any property Φ and any set A, $\{\text{standard } x \in A | \Phi(x)\}$ denotes a standard set whose standard elements are precisely those standard $x \in A$ having the property Φ. On the other hand, $\{x \in A | \Phi(x)\}$ is an external set, whose elements are precisely those external $x \in A$ having the property Φ. If Φ is an internalization of a standard property and A is internal, then $\{\text{internal } x \in A | \Phi(x)\}$ is the internal set of all internal $x \in A$ having the property Φ.

As a first example of the role played by the Principle of Standardization, we prove a theorem which can be used to justify proofs by induction for any statement about natural numbers (not only a standard one).

THEOREM 1. *If A is a set such that*
(i) $0 \in A$, *and*
(ii) *for all standard natural numbers n, if $n \in A$ then $(n+1) \in A$,*
then A contains all standard natural numbers (i.e., $^\circ N \subseteq A$).

Notice that we cannot conclude that $N \subseteq A$, unless A is standard.

Proof. Let us assume that $^\circ N \setminus A \neq \varnothing$. Since all elements of $^\circ N \setminus A$ are standard, this implies that the standard set $^*({}^\circ N \setminus A) \neq \varnothing$. Every nonempty standard set of natural numbers has a standard least element. Since $^\circ N \setminus A$ and $^*({}^\circ N \setminus A)$ have the same standard elements, the standard least element, k, of $^*({}^\circ N \setminus A)$ is also the least element of $^\circ N \setminus A$. If $k = 0$, we have a contradiction with (i), and if $k = l + 1$, then $l \in A$ and we have a contradiction with (ii).■

The next theorem establishes a simple relationship between standard and external concepts of finiteness and natural number. In particular, it implies that all elements of a standard finite set are standard.

THEOREM 2. *An external set is standard finite if and only if it is external finite and all of its elements are standard. The standard natural numbers coincide with the external natural numbers. Standard algebraic operations on standard natural numbers coincide with the corresponding external operations.*

Proof. We proceed to prove that every standard finite set X is external finite and has only standard elements, by induction on the standard number of elements of X, $\|X\|$. The statement is clearly true if $\|X\|=0$; so assume that $\|X\|=n+1$ where n is standard, and the statement is true for all standard sets with n elements. We write $X=(X\setminus\{x\})\cup\{x\}$ where x is some standard element of $X(X\neq\varnothing)$. Since $\|X\setminus\{x\}\|=n$, $X\setminus\{x\}$ is external finite and all its elements are standard, by the inductive assumption. The set $\{x\}$ is also external finite, and its single element is standard. We conclude that X is external finite (the union of two finite sets is finite) and all elements of X are standard, thus proving the statement for all standard sets with $n+1$ elements. Theorem 1 then shows that it holds for all standard finite sets.

Conversely, let X be an external finite set, all elements of which are standard; we prove that X is a standard finite set by external induction on the external number of elements of X, ${}^{\mathcal{E}}\|X\|$ (the Principle of Induction is of course a consequence of the axioms of Zermelo set theory, and thus holds in the external universe). The case ${}^{\mathcal{E}}\|X\|={}^{\mathcal{E}}0$ is again clear, so we proceed with the inductive step. If ${}^{\mathcal{E}}\|X\|={}^{\mathcal{E}}n^{\mathcal{E}}+{}^{\mathcal{E}}1$, we write $X=(X^{\mathcal{E}}\setminus{}^{\mathcal{E}}\{x\})^{\mathcal{E}}\cup{}^{\mathcal{E}}\{x\}=(X\setminus\{x\})\cup\{x\}$ (for some $x\in X$) and conclude that $X\setminus\{x\}$ is standard finite by the inductive assumption, and $\{x\}$ is standard finite, because x is standard. Therefore X is standard finite and we are done.

The proofs of the remaining claims are similar, and we omit them. Induction based on Theorem 1 should be used in one direction, and external induction in the other direction.■

Integers (rationals, respectively) are usually defined as certain ordered pairs of natural numbers (integers, respectively). Since external ordered pairs coincide with the standard ones for standard objects, it is not surprising that Theorem 2 holds for integers and rational numbers as well; the proof is trivial but tedious, and we again omit it. The relationship between standard and external real numbers is examined in §4.

We conclude with a restatement of the Principle of Saturation in a stronger form. In place of a standard set A of standard parameters, we allow any external set of internal parameters, as long as it is not too large. We say that an external set A has a *standard size*, if its elements can be enumerated by standard elements of some standard set; more precisely, if there is a standard set B and an external one-to-one mapping of ${}^\circ B$ onto $A(\|B\|$ is then called the *size* of A).

THE STRONG PRINCIPLE OF SATURATION. *Let Φ be a standard property. If A is a set of standard size, all elements of which are internal, and if for every external finite $a\subseteq A$ there is an internal y such that $\Phi^{\mathcal{G}}(x,y)$ holds simultaneously for all $x\in a$, then there is an internal y such that $\Phi^{\mathcal{G}}(x,y)$ holds simultaneously for all $x\in A$.*

In most applications, one uses the following corollary:

THEOREM 3 (The Principle of Extension). *Let A be a standard set. If f is an external function defined on ${}^\circ A$ and with internal values, then there is an internal function $F\supseteq f$.*

Proof. (The graph of) f is a set of internal objects having a standard size, as witnessed by the mapping which assigns to each $x\in{}^\circ A$ the pair $(x,f(x))$. If $a\subseteq f$ is external finite, then a is (a graph of) a function, whose domain is an external finite set of standard elements, and all values of which are internal. It follows by Theorem 2, that $\mathrm{dom}(a)$ is a standard finite set, and then, by induction based on Theorem 1, that a is internal. Therefore, $y=a$ is an internal set such that "y is a function and $x\in y$" holds simultaneously for all $x\in a$. We conclude that there is an internal F for which "F is a function and $x\in F$" holds simultaneously for all $x\in f$. But then F is an internal function and $F\supseteq f$.■

This completes description of our universe of discourse. The next three paragraphs present samples of nonstandard arguments in our system. From now on, we adopt the following notational convention: Unless explicitly stated otherwise, lightface letters denote standard objects, Greek letters denote internal objects, and boldface letters stand for arbitrary (external) objects.

4. The real numbers. Let R be the standard set of real numbers; we know already that it has nonstandard elements (since it is infinite). Our first goal is to classify elements of R. We say that an internal real number ρ is *infinitely large* (or just *infinite*) if $|\rho| \geqslant n$ for every standard natural number n. Otherwise, it is called *finitely large* (or just *finite*). We say that ρ is *infinitely small* (or *infinitesimal*) if $|\rho| \leqslant 1/n$ for every standard natural number $n \neq 0$.

THEOREM 1. *Every nonstandard natural number is infinitely large.*

Proof. Let $\nu \in N \setminus {}^\circ N$ and $n \in {}^\circ N$; since $n = \{0, 1, \ldots, n-1\}$ is a standard finite set, all of its elements are standard (Theorem 2 in §3) and $\nu \notin \{0, 1, \ldots, n-1\}$. By linearity of $<$, $\nu \geqslant n$.■

We see that infinitely large real numbers exist. If ρ is infinitely large, i.e., $|\rho| \geqslant n$ for all $n \in {}^\circ N$, then $|1/\rho| < 1/n$ for all $n \in {}^\circ N$, and $1/\rho$ is infinitesimal. The only standard infinitesimal is 0; there are no standard infinitely large reals (Archimedean property). 1 and $1 + (1/\nu)$, for ν infinitely large, are examples of finite, but not infinitely small, real numbers.

For ρ, $\sigma \in R$, we write $\rho \approx \sigma$, and say that ρ is *infinitely close* to σ, if $\rho - \sigma$ is infinitesimal. Clearly, $\approx$ is an equivalence relation on R; it is good to know that $\approx$ is not internal.

Proof. Otherwise, $\mathbf{I} = \{\text{internal } \sigma \in R \mid \sigma \approx 0\}$ (the set of all infinitesimals) and then also $N \cap \{\text{internal } \rho \in R \mid \rho = 1/\sigma \text{ for some } \sigma \in \mathbf{I},\ \sigma \neq 0\}$ would be internal; the latter set, however, is exactly $N \setminus {}^\circ N$, which we proved not to be internal.■

The *monad* of $\rho \in R$, $\mathbf{M}(\rho)$, is defined as the set $\{\sigma \in R \mid \sigma \approx \rho\}$; again, $\mathbf{M}(\rho)$ is not internal for any $\rho \in R$ (otherwise, $\mathbf{I} = \mathbf{M}(0) = \{\sigma - \rho \mid \sigma \in \mathbf{M}(\rho)\}$ would be internal).

A very important tool in Nonstandard Analysis is the mapping st, which assigns to each finite real number a standard real number infinitely close to it.

THEOREM 2. *For each finitely large $\rho \in R$ there is a unique standard $r \in R$ such that $\rho \approx r$ (i.e., $\rho \in \mathbf{M}(r)$); we denote this r by* st (ρ) *and call it the standard part of ρ.*

Notice that dom(st) is the noninternal set of all finite reals, ran(st) is the noninternal set ${}^\circ R$ of all standard reals, and $\mathrm{st}^{-1}(r) = \mathbf{M}(r)$ for all $r \in {}^\circ R$. In particular, st is not internal.

Proof. Since ρ is finite, $\rho \leqslant n$ for some $n \in {}^\circ N$. The standard set $X = \{\text{standard } r \in R \mid r \leqslant \rho\}$ is bounded from above by n, and therefore has a supremum. Let $r = \sup X$; we prove that $r \approx \rho$. But if not, then there is $n \in {}^\circ N$ such that $|r - \rho| \geqslant 1/n$, i.e., either $r - (1/n) \geqslant \rho$, or $r + (1/n) \leqslant \rho$. Both possibilities contradict the definition of r. To prove uniqueness, we note that $r \approx \rho$ and $s \approx \rho$ imply $r \approx s$, that is, $(r - s)$ infinitesimal. Since 0 is the only standard infinitesimal, we have $r - s = 0$.■

Some simple properties of the function st are needed later.

THEOREM 3. *If A, B and $\langle A_i \mid i \in I \rangle$ are standard, then*

$$\mathrm{st}^{-1}\Big(\bigcup_{i \in I} A_i\Big) = \bigcup_{i \in {}^\circ I} \mathrm{st}^{-1}(A_i);$$

$$\mathrm{st}^{-1}\Big(\bigcap_{i \in I} A_i\Big) = \bigcap_{i \in {}^\circ I} \mathrm{st}^{-1}(A_i);$$

$$\mathrm{st}^{-1}(A \setminus B) = \mathrm{st}^{-1}(A) \setminus \mathrm{st}^{-1}(B).$$

Proof. We show only $\bigcap_{i \in {}^\circ I} \mathrm{st}^{-1}(A_i) \subseteq \mathrm{st}^{-1}(\bigcap_{i \in I} A_i)$; the other cases are similar. If $\rho \in \bigcap_{i \in {}^\circ I} \mathrm{st}^{-1}(A_i)$, then $r = \mathrm{st}(\rho) \in A_i$ for all standard $i \in I$. Since r is standard, it follows that $r \in \bigcap_{i \in I} A_i$ and therefore $\rho \in \mathrm{st}^{-1}(\bigcap_{i \in I} A_i)$.■

The relationship between standard and external rational numbers is discussed in §3; we see there that x is a standard rational if and only if it is an external rational. However, from the standard point of view, the set of all rationals is $Q=\{\text{standard } x|x \text{ is a rational}\}$, while externally it is ${}^{\mathcal{E}}Q={}^{\circ}Q=\{x|x \text{ is a standard rational}\}$. Because of this, the standard and the external real numbers do not exactly coincide; nevertheless, the relationship between the two concepts is very simple. To be specific, we identify real numbers with the lower classes of the corresponding Dedekind cuts; if then r is a standard real number, $r=\{\text{standard } q\in Q|q\leqslant r\}$, then ${}^{\circ}r=\{q\in r|q \text{ is standard}\}\subseteq{}^{\circ}Q$ is an external real number such that, for all standard rational q, $q\leqslant r$ if and only if $q^{\mathcal{E}}\leqslant{}^{\circ}r$. Conversely, if $\mathbf{r}=\{\mathbf{q}\in{}^{\mathcal{E}}Q|\mathbf{q}^{\mathcal{E}}\leqslant\mathbf{r}\}$ is an external real number, then ${}^{*}\mathbf{r}\subseteq Q$ is a standard real number such that, for all standard rational q, $q^{\mathcal{E}}\leqslant\mathbf{r}$ if and only if $q\leqslant{}^{*}\mathbf{r}$. The correspondence via $^{\circ}$ and * is easily seen to preserve algebraic operations and other algebraic and analytic properties of real numbers; for example, if A is a standard set of real numbers and $\mathbf{A}=\{{}^{\circ}r|r\in A\}$ is the corresponding set of external real numbers, ${}^{\circ}(\sup(A))={}^{\mathcal{E}}\sup(\mathbf{A})$. Since we are not going to be concerned with properties of reals as sets (Dedekind cuts), but merely with their algebraic properties, we identify the standard reals (i.e., the elements of ${}^{\circ}R$) with the external reals (i.e., the elements of ${}^{\mathcal{E}}R$), and ${}^{\circ}R$ (but, of course, not R) with ${}^{\mathcal{E}}R$. This convention is notationally convenient in §6.

5. Some general topological theorems. In this paragraph, we look at several basic topological notions from the nonstandard point of view. Let T be a standard Hausdorff topological space. For each standard $t\in T$, we define the *monad* of t, $\mathbf{M}(t)$, as the set of all $\tau\in T$ which belong into every standard neighborhood of t; i.e., $\mathbf{M}(t)=\cap\{U|U \text{ is a standard neighborhood of } t\}$. We say that elements of $\mathbf{M}(t)$ are *infinitely close* to t, and write $\tau\approx t$, when $\tau\in\mathbf{M}(t)$. Notice that we do not define monads of, and infinite closeness for, nonstandard elements of T; this can be done only if the topology of T is determined by a uniformity. A point $\tau\in T$ is called *near-standard* if $\tau\in\mathbf{M}(t)$ for some standard $t\in T$. We note that t is uniquely determined in Hausdorff spaces ($t\neq t'$ implies existence of neighborhoods U, U' such that $t\in U$, $t'\in U'$ and $U\cap U'=\varnothing$; but $\mathbf{M}(t)\subseteq U$, $\mathbf{M}(t')\subseteq U'$); we denote this t by $\mathrm{st}(\tau)$. In this way, st is an external mapping of near-standard points of T onto the standard points of T; it is easy to check that it has all properties proved in §4 for the special case $T=R$.

We now prove several typical results.

THEOREM 1. *A standard sequence $\langle t_n|n\in N\rangle$ of elements of T converges to $t\in T$ if and only if $t_\nu\approx t$ for all infinitely large $\nu\in N$.*

Proof. (1) Assume that $\lim_{n\to\infty}t_n=t$. If U is any standard neighborhood of t, then $t_n\in U$ for all $n\geqslant n_U$ where $n_U\in{}^{\circ}N$; in particular, $t_\nu\in U$ for all infinitely large $\nu\in N$. Thus $t_\nu\approx t$ for any such ν.

(2) Assume, conversely, that $t_\nu\approx t$ for all infinitely large $\nu\in N$. Let U be a standard neighborhood of t; then there is an internal $\nu_U\in N$ such that $t_\nu\in U$ for all internal $\nu\geqslant\nu_U$ (let ν_U be any infinitely large integer). By the Principle of Embedding, there is a standard $n_U\in N$ such that $t_n\in U$ for all standard $n\geqslant n_U$, and we have $\lim_{n\to\infty}t_n=t$.■

THEOREM 2. *Let $f: T\to R$ be a standard function, and let $t\in T$ be standard. Then f is continuous at t if and only if for all $\tau\in T$, $\tau\approx t$ implies $f(\tau)\approx f(t)$ (in other words, $f[\mathbf{M}(t)]\subseteq\mathbf{M}(f(t))$).*

Proof. (1) Assume that for any standard $\varepsilon>0$ there is a standard neighborhood U of t such that, for all $x\in T$, $x\in U$ implies $|f(x)-f(t)|<\varepsilon$. If $\tau\approx t$, then $\tau\in U$ for all standard U, and hence $|f(\tau)-f(t)|<\varepsilon$ for all standard $\varepsilon>0$. We see that $f(\tau)\approx f(t)$.

(2) Assume, conversely, that $\tau\approx t$ implies $f(\tau)\approx f(t)$ for all $\tau\in T$. Working toward a contradiction, let $\varepsilon>0$ be standard and such that for any standard neighborhood U of t there is $x\in U$ for which $|f(x)-f(t)|\geqslant\varepsilon$. Then the standard property "$x\in U$ and $|f(x)-f(t)|\geqslant\varepsilon$" can be satisfied simultaneously for any standard finite set $\{U_1,\ldots,U_n\}$ of neighborhoods of t (because $\cap_{1\leqslant i\leqslant n}U_i$

is also a neighborhood of t), and the Weak Principle of Saturation provides $\tau \in T$ satisfying "$\tau \in U$ and $|f(\tau)-f(t)| \geqslant \varepsilon$" simultaneously for all such standard U. But then $\tau \approx t$ and not $f(\tau) \approx f(t)$, contradicting the assumption.■

Yet another argument in a similar vein establishes

THEOREM 3. *A standard set $A \subseteq T$ is open if and only if $\mathbf{M}(t) \subseteq A$ for all standard $t \in A$. A is closed if and only if, for any standard $t \in T$, $\mathbf{M}(t) \cap A \neq \varnothing$ implies $t \in A$,*

It is a little harder to obtain a characterization of compactness.

THEOREM 4. *A standard closed set $A \subseteq T$ is compact if and only if all elements of A are near-standard.*

Proof. (1) Assume that A is compact. Let $\tau \in A$; we show that $\tau \approx t$ for some $t \in A$. If not, then each standard $t \in A$ has a standard neighborhood U_t such that $\tau \notin U_t$. The system {standard $U_t | t \in A$ and t is standard} is obviously a standard cover of A, and therefore has a standard finite subcover, say $\{U_{t_1}, \ldots, U_{t_n}\}$. This means that $A \subseteq \bigcup_{1 \leqslant i \leqslant n} U_{t_i}$ and implies $\tau \in \bigcup_{1 \leqslant i \leqslant n} U_{t_i}$. Since all elements of the standard finite set $\{1,2,\ldots,n\}$ are standard, we conclude that $\tau \in U_{t_i}$ for some standard $t_i \in A$, contradicting the choice of U_{t_i}.

(2) Assume, conversely, that for each $\tau \in A$ there is a standard $t \in T$ such that $t \approx \tau$; since A is closed, $t = \mathrm{st}(\tau) \in A$ by Theorem 3. Working towards a contradiction, we assume that $\langle U_i | i \in I \rangle$ is a standard cover of A by open sets such that, for each standard finite $I_0 \subseteq I$, there is $x \in A \setminus U_i$ for all $i \in I_0$. By the Weak Principle of Saturation, we then can find $\xi \in T$ such that $\xi \in A \setminus U_i$ for all standard $i \in I$. But $\xi \in A$ is near-standard and $A \setminus U_i$ is a closed set; so Theorem 3 implies $x = \mathrm{st}(\xi) \in A \setminus U_i$, for all standard $i \in I$. This is a contradiction with the assumption that $\langle U_i | i \in I \rangle$ is a standard cover of A.■

As a last example, we give a nonstandard proof of a standard theorem.

THEOREM 5. *An image of a compact set by a continuous function is compact.*

Proof. Let $f: T \to R$ be a standard continuous function, and let $A \subseteq T$ be a standard compact set. According to Theorems 3 and 4, it suffices to show that, for all $\tau \in A$, $f(\tau)$ is near-standard and $\mathrm{st}(f(\tau)) \in f[A]$. However, since A is compact, $\tau \in A$ implies $\tau \approx t \in A$. By the continuity of f, we then have $f(\tau) \approx f(t) \in f[A]$ (see Theorem 2), so $f(\tau)$ is indeed near-standard and $\mathrm{st}(f(\tau)) = f(t) \in f[A]$.■

6. Topics in nonstandard analysis. Good expositions of the nonstandard approach to Calculus abound in the published literature; we recommend, in particular, Keisler [5]. Also, no methods essentially different from those illustrated by the previous examples are called for. For these reasons we merely hint at the flavor of the subject by formulating a definition of the derivative, and then devote the rest of this section to nonstandard measure theory. This topic is technically much more interesting, because of its heavy use of external concepts.

Let α, β, $\iota \in R$, $\iota \neq 0$. We say that α is *infinitely close to β relative to ι* (notation: $\alpha \approx \beta(\iota)$) if $(\alpha - \beta)/\iota$ is infinitesimal. In particular then $\alpha \approx \beta$ if and only if $\alpha \approx \beta$ (1). Let f be a function from an interval I into R, and let $a \in I$. (All lightface sets are standard.) We say that $A \in R$ is the *derivative* of f at a if $\Delta f = f(a+\iota) - f(a)$ is infinitely close to $df = A \cdot \iota$ relative to ι, for all infinitesimal $\iota \neq 0$. We then set

$$f'(a) = A = \mathrm{st}\left(\frac{f(a+\iota) - f(a)}{\iota}\right).$$

A simple proof in the spirit of §5 shows that this definition agrees with the usual notion of derivative.

We now turn to nonstandard measure theory. The basic idea of the nonstandard approach, implicit already in Robinson [11], and developed explicitly in various different ways by Bernstein, Wattenberg, Henson, Loeb, Anderson, and others (see [4] for references), is to approximate standard measures "infinitely closely" by internal measures on internal *finite* sets. Measures on finite sets are very simple objects: Each internal measure μ on an internal finite set Ω is completely determined by its point-mass function $\phi_\mu:\Omega\to R$ where $\phi_\mu(\omega)=\mu(\{\omega\})$ for all $\omega\in\Omega$. Conversely, every internal $\phi:\Omega\to R$ determines a unique measure μ_ϕ on (the algebra of all internal subsets of) Ω by $\mu_\phi(X)=\Sigma_{\omega\in X}\phi(\omega)$, for all internal $X\subseteq\Omega$. The simplest and most important example of a measure on a finite set Ω is the *counting measure* μ_c defined by $\mu_c(X)=\|X\|/\|\Omega\|$ for all internal $X\subseteq\Omega$; then of course $\phi_{\mu_c}(\omega)=1/\|\Omega\|$ for all $\omega\in\Omega$. (Here, as in §3, $\|X\|$ denotes the internal number of elements of the internal finite set X.) The approximation procedure then makes it possible to substitute, for example, the counting measure on an internal finite set for the standard Lebesgue measure.

We say that (X, S, m) is a *measure space* if S is an algebra of subsets of X and m is a finitely additive nonnegative real-valued measure on S (we do not assume σ-completeness of S and σ-additivity of m unless explicitly specified). The following two theorems describe one way the previously mentioned approximation procedure can be realized.

THEOREM 1. *For every standard measure space (X, S, m) there is an internal measure space $(\Omega, \mathcal{P}(\Omega), \mu)$ such that ${}^\circ X\subseteq\Omega\subseteq X$, Ω is finite, and $m(A)=\mu(A\cap\Omega)$ for all standard $A\in S$.*

Proof. For any standard finite subsets, $\{x_1,\dots,x_m\}$ and $\{A_1,\dots,A_n\}$, of X and S, respectively, there are a standard finite $\sigma\subseteq A$ and a standard function $f:\sigma\to R$ such that "$x_i\in\sigma$ and $m(A_i)=\Sigma_{x\in A_i\cap\sigma}f(x)$" holds for all $i=1,\dots,m$ and $j=1,\dots,n$.

To see this, we consider the partition of X into 2^n sets of the form $\tilde{A}_1\cap\cdots\cap\tilde{A}_n$, where each $\tilde{A}_j$ is either A_j or $X\setminus A_j$. Whenever $\tilde{A}_1\cap\cdots\cap\tilde{A}_n\neq\varnothing$, we choose one standard $y\in\tilde{A}_1\cap\cdots\cap\tilde{A}_n$, put y into σ, and set $f(y)=m(\tilde{A}_1\cap\cdots\cap\tilde{A}_n)$. If $x_i(i=1,\dots,m)$ is not one of the y's chosen, we put it into σ also, and then set $f(x_i)=0$. It is easy to check that σ and f work.

An application of the Weak Principle of Saturation now yields an internal finite $\Omega\subseteq A$ and an internal function $\phi:\Omega\to R$ such that $x\in\Omega$ holds for all $x\in{}^\circ A$ and $m(A)=\Sigma_{\xi\in A\cap\Omega}\phi(\xi)$ holds for all standard $A\in S$. The measure μ_ϕ determined on $\mathcal{P}(\Omega)$ by the point-mass function ϕ has the required properties.■

THEOREM 2. *For every internal measure space $(\Omega, \mathcal{P}(\Omega), \mu)$ where $\mu(\Omega)$ is finitely large, and ${}^\circ X\subseteq\Omega\subseteq X$, there is a standard measure space $(X, \mathcal{P}(X), m)$ such that $m(A)\approx\mu(A\cap\Omega)$ for all standard $A\subseteq X$.*

Proof. The function $\mathbf{m}$ defined for standard $A\subseteq X$ by $\mathbf{m}(A)=\mathrm{st}(\mu(A\cap\Omega))$ is external. However, $m={}^*\mathbf{m}$ is a standard set having exactly the same standard elements as $\mathbf{m}$; in particular, it is a function, and $m(A)=\mathbf{m}(A)\approx\mu(A\cap\Omega)$ for all $A\in\mathcal{P}(X)$. Finite additivity follows easily from the fact that the sum of a standard finite set of infinitesimals is infinitesimal (induction based on Theorem 1 in §3).■

It is easy to check, using approximations by simple functions, that Theorem 1 (Theorem 2, respectively) implies, for all standard bounded m-measurable (μ-measurable, respectively) functions $f:X\to R$ that $\int f dm\approx\int f\restriction\Omega d\mu$ where, for Ω internally finite, $\int f\restriction\Omega d\mu=\Sigma_{\omega\in\Omega}f(\omega)\phi_\mu(\omega)$. (As usual, a standard function $f:X\to R$ is *bounded* if $|f(x)|\leqslant K$ for some $K\in R$ and all $x\in X$.)

Another immediate corollary of the preceding theorems is the fact that every standard finitely additive measure on some algebra of subsets of A can be extended to a standard finitely additive measure on the standard algebra of all subsets of A. For example, starting with the measure m_1, defined on the algebra generated by left-closed, right-open subintervals of [0, 1), and assigning to each interval its length, we use Theorem 1 to find its internal approximation μ, and then

Theorem 2 to find a standard approximation m_2 of μ, defined on all subsets of [0, 1); clearly, $m_1 \subseteq m_2$. Unfortunately, this approach does not seem to give us any clue as to which subsets of [0, 1) are Lebesgue measurable, or guarantee that m_2 is equal to the Lebesgue measure on such sets (or, for that matter, is even σ-additive).

The crucial new idea, introduced by Loeb in [6], is to try to recover $m(A)$ from μ using $\mathrm{st}^{-1}(A) \cap \Omega$, rather than $A \cap \Omega$, as we do in Theorem 2. (To see the difference, note, e.g., that all nonzero infinitesimals belong to $\mathrm{st}^{-1}(N) \setminus N$, while all infinitely large natural numbers belong to $N \setminus \mathrm{st}^{-1}(N)$.) However, the set $\mathrm{st}^{-1}(A)$ is usually not internal, and we are thus faced with a problem of extending the measure μ to external sets. Loeb at this point relies on Carathéodory's Extension Theorem (see [12]); but it is instructive and not too difficult to proceed directly. This is what we now begin to do.

From now on, let (Ω, Σ, μ) be a fixed internal measure space where $\mu(\Omega)$ is finite. Ideally, we might want to extend μ itself to external $\mathbf{X} \subseteq \Omega$. A necessary condition on μ is that

$$\text{for all internal } A_1, A_2 \in \Sigma, \text{ if } A_1 \subseteq \mathbf{X} \subseteq A_2, \text{ then } \mu(A_1) \leqslant \mu(\mathbf{X}) \leqslant \mu(A_2). \tag{*}$$

We could then define $\mu_-(\mathbf{X}) = {}^{\mathfrak{S}}\sup\{\mu(A) | A \in \Sigma \text{ and } A \subseteq \mathbf{X}\}$, $\mu^+(\mathbf{X}) = {}^{\mathfrak{S}}\inf\{\mu(A) | A \in \Sigma \text{ and } A \supseteq \mathbf{X}\}$, and say that $\mathbf{X}$ is measurable if $\mu_-(\mathbf{X}) = \mu^+(\mathbf{X})$. But, since not every external bounded set of internal real numbers has a supremum (consider ${}^\circ N$), the indicated ${}^{\mathfrak{S}}\sup$ and ${}^{\mathfrak{S}}\inf$ need not exist, and it would be necessary to externally complete the internal reals first. It appears that completions with reasonable algebraic properties automatically identify each finite $\alpha \in R$ with $\mathrm{st}(\alpha)$. Since we are ultimately interested in measures whose values are standard real numbers (infinitely close to the values of μ) anyway, it is simpler to take the standard part right away, and look for an external measure $\mathbf{m}$ satisfying

$$\text{for all internal } A_1, A_2 \in \Sigma, \text{ if } A_1 \subseteq \mathbf{X} \subseteq A_2, \text{ then } \mathrm{st}(\mu(A_1)) \leqslant \mathbf{m}(\mathbf{X}) \leqslant \mathrm{st}(\mu(A_2)). \tag{**}$$

(We remind the reader that we identify external real numbers with the standard ones as in §4; so external measures have standard reals as values.)

These considerations motivate the following definition: For all external sets $\mathbf{X} \subseteq \Omega$,

$$\begin{aligned} \mathbf{m}_-(\mathbf{X}) &= \sup\{\text{standard } r \in R \,|\, r = \mathrm{st}(\mu(A)) \text{ for some} \\ &\qquad A \in \Sigma \text{ such that } A \subseteq \mathbf{X}\}, \\ \mathbf{m}^+(\mathbf{X}) &= \inf\{\text{standard } r \in R \,|\, r = \mathrm{st}(\mu(A)) \text{ for some} \\ &\qquad A \in \Sigma \text{ such that } A \supseteq \mathbf{X}\}. \end{aligned}$$

We let $\Sigma^* = \{\mathbf{X} \subseteq \Omega | \mathbf{m}_-(\mathbf{X}) = \mathbf{m}^+(\mathbf{X})\}$ and set $\mathbf{m}(\mathbf{X}) = \mathbf{m}_-(\mathbf{X}) = \mathbf{m}^+(\mathbf{X})$ for $\mathbf{X} \in \Sigma^*$.

THEOREM 3. *Σ^* is an external σ-algebra of subsets of Ω, $\Sigma^* \supseteq \Sigma$, $\mathbf{m}(A) \approx \mu(A)$ for $A \in \Sigma$, $\mathbf{m}$ is an external σ-additive measure on Σ^*, and Σ^* is complete with respect to $\mathbf{m}$. Moreover, Σ^* is the smallest external set with those properties, and $\mathbf{m}$ is uniquely determined on Σ^*.*

Proof. We first notice that $\mathbf{X} \in \Sigma^*$ if and only if for every standard $\varepsilon > 0$ there exist internal $A_1, A_2 \in \Sigma$ such that $A_1 \subseteq \mathbf{X} \subseteq A_2$ and $\mu(A_2 \setminus A_1) < \varepsilon$. It is then obvious that $\Sigma \subseteq \Sigma^*$, $\mathbf{m}(A) = \mathrm{st}(\mu(A))$ for $A \in \Sigma$, and Σ^* is closed under complements, and almost equally easy to show that it is closed under unions: If $\mathbf{X}_1, \mathbf{X}_2 \in \Sigma^*$ and $\varepsilon > 0$ is standard, choose $A_{11}, A_{12}, A_{21}, A_{22} \in \Sigma$ so that

$$A_{11} \subseteq \mathbf{X}_1 \subseteq A_{12} \quad \text{and} \quad \mu(A_{12} \setminus A_{11}) < \frac{\varepsilon}{2},$$

$$A_{21} \subseteq \mathbf{X}_2 \subseteq A_{22} \quad \text{and} \quad \mu(A_{22} \setminus A_{21}) < \frac{\varepsilon}{2}.$$

Then $A_{11} \cup A_{21}$ and $A_{12} \cup A_{22}$ belong to Σ, $A_{11} \cup A_{21} \subseteq \mathbf{X}_1 \cup \mathbf{X}_2 \subseteq A_{12} \cup A_{22}$, and

$$\mu[(A_{12} \cup A_{22}) \setminus (A_{11} \cup A_{21})] \leqslant \mu(A_{12} \setminus A_{11}) + \mu(A_{22} \setminus A_{21}) < \varepsilon.$$

We next show that, for each external sequence $\langle \mathbf{X}_n | n \in {}^{\mathfrak{S}}N = {}^\circ N \rangle$ of pairwise disjoint elements of Σ^*, $\mathbf{X} = \bigcup_{n \in {}^\circ N} \mathbf{X}_n \in \Sigma^*$ and $\mathbf{m}(\mathbf{X}) = \sum_{n \in {}^\circ N} \mathbf{m}(\mathbf{X}_n)$.

For each $n \in {}^\circ N$, choose $A_{n1}, A_{n2} \in \Sigma$ such that $A_{n1} \subseteq \mathbf{X}_n \subseteq A_{n2}$ and $\mu(A_{n2} \setminus A_{n1}) < \varepsilon/2^n$. The naive attempt of letting $A_1 = \bigcup_{n \in {}^\circ N} A_{n1}$ and $A_2 = \bigcup_{n \in {}^\circ N} A_{n2}$ fails, because A_1 and A_2 need not be internal. We need a more complicated argument, using the Strong Principle of Saturation. To begin with, sets $\langle A_{n1} | n \in {}^\circ N \rangle$ are pairwise disjoint, and therefore, for any $k \in {}^\circ N$, $\Sigma_{0 \leqslant n \leqslant k} \mathbf{m}(A_{n1}) = \mathrm{st}(\mu(\bigcup_{0 \leqslant n \leqslant k} A_{n1})) \leqslant \mathrm{st}(\mu(\Omega))$. Since $\mu(\Omega)$ is finitely large, this implies that the external series $\Sigma_{n \in {}^\circ N} \mathbf{m}(A_{n1})$ converges. The inequalities

$$\begin{aligned} \sum_{n \in {}^\circ N} \mathbf{m}(\mathbf{X}_n) &\leqslant \sum_{n \in {}^\circ N} \mathrm{st}(\mu(A_{n2})) \\ &= \sum_{n \in {}^\circ N} [\mathrm{st}(\mu(A_{n1})) + \mathrm{st}(\mu(A_{n2} \setminus A_{n1}))] \\ &\leqslant \sum_{n \in {}^\circ N} \left[\mathbf{m}(A_{n1}) + \frac{\varepsilon}{2^n} \right] \leqslant \mathbf{m}(\Omega) + 2\varepsilon \end{aligned}$$

then show that $\Sigma_{n \in {}^\circ N} \mathbf{m}(\mathbf{X}_n)$ and $\Sigma_{n \in {}^\circ N} \mathbf{m}(A_{n2})$ also converge; we let $M = \Sigma_{n \in {}^\circ N} \mathbf{m}(\mathbf{X}_n)$. It then also follows from these inequalities that one can choose $k \in {}^\circ N$ so that

$$M = \sum_{n \in {}^\circ N} \mathbf{m}(\mathbf{X}_n) < \sum_{0 \leqslant n \leqslant k} \mathbf{m}(A_{n1}) + 3\varepsilon \quad \text{and} \quad \sum_{k \leqslant n \in {}^\circ N} \mathbf{m}(A_{n2}) < \varepsilon.$$

We let $A_1 = \bigcup_{0 \leqslant n \leqslant k} A_{n1}$ be the desired lower approximation to $\mathbf{X} = \bigcup_{n \in {}^\circ N} \mathbf{X}_n$; clearly, $A_1 \subseteq \mathbf{X}$ and $A_1 \in \Sigma$. To get the upper approximation, we prove the following result:

CLAIM. *There is an internal finite sequence $\langle A_{\nu 2} | \nu \leqslant \nu_0 \rangle$ $(\nu_0 \in N \setminus {}^\circ N)$ of elements of Σ extending $\langle A_{n2} | n \in {}^\circ N \rangle$ and such that $\Sigma_{k \leqslant \nu \leqslant \nu_0} \mu(A_{\nu 2}) \leqslant \varepsilon$.*

We then let $A_2 = \bigcup_{\nu \leqslant \nu_0} A_{\nu 2}$; $A_2 \in \Sigma$, because Σ is an internal algebra and is thus closed under internal finite unions, and $\mathbf{X}_n \subseteq A_{n2}$ for all $n \in {}^\circ N$, so that $\mathbf{X} = \bigcup_{n \in {}^\circ N} \mathbf{X}_n \subseteq \bigcup_{n \in {}^\circ N} A_{n2} \subseteq \bigcup_{\nu \leqslant \nu_0} A_{\nu 2} = A_2$. Moreover, $\mu(A_2 \setminus A_1) \leqslant \mu[\bigcup_{0 \leqslant n \leqslant k} \mu(A_{n2} \setminus A_{n1}) \cup \bigcup_{k \leqslant \nu \leqslant \nu_0} A_{\nu 2}] \leqslant \Sigma_{0 \leqslant n \leqslant k} \mu(A_{n2} \setminus A_{n1}) + \Sigma_{k \leqslant \nu \leqslant \nu_0} \mu(A_{\nu 2}) \leqslant (\Sigma_{0 \leqslant n \leqslant k}(\varepsilon/2^n)) + \varepsilon < 3\varepsilon$. Since ε is an arbitrary standard positive number, this shows $\mathbf{X} \in \Sigma^*$. Furthermore,

$$\begin{aligned} \sum_{n \in {}^\circ N} \mathbf{m}(\mathbf{X}_n) - 3\varepsilon &< \sum_{0 \leqslant n \leqslant k} \mathbf{m}(A_{n1}) \leqslant \mathbf{m}(A_1) \leqslant \mathbf{m}(\mathbf{X}) \leqslant \mathbf{m}(A_2) \\ &\leqslant \mathbf{m}(A_1) + \mathbf{m}(A_2 \setminus A_1) \leqslant \sum_{n \in {}^\circ N} \mathbf{m}(\mathbf{X}_n) + 3\varepsilon, \end{aligned}$$

being true for all standard $\varepsilon > 0$, implies

$$\mathbf{m}\left(\bigcup_{n \in {}^\circ N} \mathbf{X}_n \right) = \sum_{n \in {}^\circ N} \mathbf{m}(\mathbf{X}_n).$$

To prove the Claim, we first use the Principle of Extension (Theorem 3 of §3) to find an internal function $\langle A_{\nu 2} | \nu \in \Delta \rangle$ such that ${}^\circ N \subseteq \Delta$ and $A_{\nu 2} = A_{n2}$ whenever $\nu = n \in {}^\circ N$. Let then $\Delta' = \{\text{internal } \nu \in \Delta | \nu \in N, A_{\nu' 2} \in \Sigma \text{ for all internal natural numbers } \nu' \leqslant \nu, \text{ and } \Sigma_{k \leqslant \nu' \leqslant \nu} \mu(A_{\nu' 2}) \leqslant \varepsilon\}$. We note that Δ' is an internal set and that ${}^\circ N \subseteq \Delta' \subseteq N$. Since ${}^\circ N$ is not internal, there exists $\nu_0 \in \Delta' \setminus {}^\circ N$; $\langle A_{\nu 2} | \nu \leqslant \nu_0 \rangle$ then has all properties required by the Claim.

We now know that $\mathbf{m}$ is an external σ-additive measure on Σ^*. If $\mathbf{X} \in \Sigma^*$ and $\mathbf{m}(\mathbf{X}) = 0$, then $\inf\{\text{standard } r \in R | r = \mathrm{st}(\mu(A)) \text{ for some } A \in \Sigma \text{ such that } A \supseteq \mathbf{X}\} = 0$, so clearly $\inf\{\text{standard } r \in R | r = \mathrm{st}(\mu(A)) \text{ for some } A \in \Sigma \text{ such that } A \supseteq \mathbf{Y}\} = 0$ for any $\mathbf{Y} \subseteq \mathbf{X}$, showing $\mathbf{Y} \in \Sigma^*$ and $\mathbf{m}(\mathbf{Y}) = 0$. This means that Σ^* is complete with respect to $\mathbf{m}$.

To prove minimality of Σ^*, let Σ^{**} be another algebra with the properties stated in Theorem 3 (for Σ^*). If $\mathbf{X} \in \Sigma^*$, then, for each $n \in {}^\circ N$, there exist $A_{n1}, A_{n2} \in \Sigma$ such that $A_{n1} \subseteq \mathbf{X} \subseteq A_{n2}$ and $\mathbf{m}(A_{n2} \setminus A_{n1}) < 1/n$. Since $\Sigma \subseteq \Sigma^{**}$ and Σ^{**} is an external σ-algebra, $A_1 = \bigcup_{n \in {}^\circ N} A_{n1}$ and $A_2 = \bigcap_{n \in {}^\circ N} A_{n2}$ both belong to Σ^{**}; moreover, $A_1 \subseteq \mathbf{X} \subseteq A_2$ and $\mathbf{m}(A_2 \setminus A_1) \leqslant \mathbf{m}(A_{n2} \setminus A_{n1}) < 1/n$ for all standard n. But $\mathbf{m}(A_2 \setminus A_1)$ is standard, so $\mathbf{m}(A_2 \setminus A_1) = 0$ and, by completeness of Σ^{**},

$\mathbf{X}\setminus A_1 \subseteq A_2 \setminus A_1$ belongs to Σ^{**}. Finally, $\mathbf{X}=(\mathbf{X}\setminus A_1)\cup A_1 \in \Sigma^{**}$. We see that $\Sigma^* \subseteq \Sigma^{**}$. In the process, we also showed that $\mathbf{m}(\mathbf{X})=\mathbf{m}(A_1)=\sup\{\text{standard } r\in R \mid r=\text{st}(\mu(A_{n1}))$ for some $n\in {}^\circ N\}$ is uniquely determined.■

Again, the usual procedure of approximating by simple functions gives the following characterization of measurability and integral (which can be used as a definition of these concepts even prior to introduction of the notion of measure).

THEOREM 4. *An external bounded function* $\mathbf{f}:\Omega\to R$ *is* $\mathbf{m}$*-measurable if and only if*

$$\sup\left\{\text{standard } r\in R \mid r=\text{st}\left(\sum_{\omega\in\Omega}\phi(\omega)\right) \textit{ for some internal } \phi \textit{ such that } \phi(\omega)\leqslant \mathbf{f}(\omega) \textit{ for all } \omega\in\Omega\right\} =$$

$$\inf\left\{\text{standard } r\in R \mid r=\text{st}\left(\sum_{\omega\in\Omega}\phi(\omega)\right) \textit{ for some internal } \phi \textit{ such that } \phi(\omega)\geqslant \mathbf{f}(\omega) \textit{ for all } \omega\in\Omega\right\}.$$

If $\mathbf{f}$ *is* $\mathbf{m}$*-measurable, the common value is equal to* $\int \mathbf{f}d\mathbf{m}$. *All internal bounded functions* $\mathbf{f}$ *are* $\mathbf{m}$*-measurable and* $\int \mathbf{f}d\mathbf{m}-\text{st}(\int \mathbf{f}d\mu)$ *for such* $\mathbf{f}$.

Although we restrict ourselves to finite measures for the sake of simplicity, conventional methods can be used to extend the construction of $\mathbf{m}$ to measure spaces where $\mu(\Omega)$ is not necessarily finite. One defines first a set Σ' by letting $\mathbf{X}\in\Sigma'$ if and only if $\mathbf{X}\subseteq A$ for some internal $A\in\Sigma$ such that $\mu(A)$ is finite, and $\sup\{\text{standard } r\in R \mid r=\text{st}(\mu(A))$ for $A\in\Sigma$ such that $A\subseteq\mathbf{X}\}=\inf\{\text{standard } r\in R \mid r=\text{st}(\mu(A))$ for $A\in\Sigma$ such that $A\supseteq\mathbf{X}\}$. One then lets $\mathbf{X}\in\Sigma^*$ if and only if $\mathbf{X}\cap A\in\Sigma'$ for all $A\in\Sigma$ such that $\mu(A)$ is finite, and sets

$$\mathbf{m}(\mathbf{X})=\sup\{\text{standard } r\in R \mid r=\text{st}(\mu(\mathbf{X}\cap A)) \text{ for } A\in\Sigma \text{ such that } \mu(A) \text{ is finite}\}, \tag{+}$$

where it is understood that $\mathbf{m}(\mathbf{X})=+\infty$ if the set in (+) is unbounded. Arguments similar to those used in proof of Theorem 3 then show that Σ^* has all properties listed in Theorem 3, except perhaps minimality.

Our stated purpose is to obtain simple nonstandard "infinitely close" approximations of standard measures. We now return to this question and show how it can be done, using Theorem 3, in case of the Lebesgue measure on the interval [0, 1).

Let ν be an infinitely large natural number, and let $\Omega=\{\text{internal } \rho\in R \mid \rho=\alpha/\nu$ for some $\alpha\in N$ such that $0\leqslant\alpha<\nu\}$. Σ is the algebra of all internal subsets of Ω, and μ is the counting measure on Ω, that is, $\mu(A)=\|A\cap\Omega\|/\nu$ for all $A\in\Sigma$. Theorem 3 supplies an external σ-algebra $\Sigma^*\supseteq\Sigma$ and an external σ-additive measure $\mathbf{m}$ on Σ^*. We then let $S=\{\text{standard } X\subseteq[0, 1) \mid \text{st}^{-1}(X)\cap\Omega\in\Sigma^*\}$ and define a standard function n on S by the requirement that $n(X)=\mathbf{m}(\text{st}^{-1}(X)\cap\Omega)$ for all $X\in S$ (existence of such n follows from the Principle of Standardization). The simple properties of st listed in Theorem 3 of §4 immediately show that S is a standard σ-algebra of subsets of [0, 1), n is a standard σ-additive measure on S, and S is complete with respect to n. We also have $n([0,x))=x$ for all standard x, $0<x<1$. Indeed, we first see that

$$\mathbf{m}([0,x)\cap\Omega)=\text{st}(\mu([0,x)\cap\Omega))=\text{st}\left(\frac{[x\nu]-1}{\nu}\right)=x,$$

because $([x\nu]-1)/\nu<x\leqslant[x\nu]/\nu$ and $1/\nu$ is infinitesimal. (Here $[x\nu]$ is the largest integer $\leqslant x\nu$.) Next, for any $x'<x$, $[0,x')\cap\Omega\subseteq\text{st}^{-1}([0,x))\cap\Omega\subseteq[0,x)\cap\Omega$, so that $x'\leqslant n([0,x))\leqslant x$ for any $x'<x$,

and thus $n([0,x))=x$. Therefore, S contains all Lebesgue measurable sets and n agrees with the Lebesgue measure on such sets. It is also evident that n is translation-invariant. Actually, it is possible to prove that S contains exactly all Lebesgue measurable sets (this result is due to Henson). Furthermore, a standard function $f:[0, 1)\to R$ is n-measurable or n-integrable if and only if $(f\circ \mathrm{st})\restriction\Omega$ is $\mathbf{m}$-measurable or $\mathbf{m}$-integrable, and $\int f\mathrm{d}n=\int(f\circ\mathrm{st})\restriction\Omega d\mathbf{m}$ whenever either side exists. (This is again proved using approximations by simple functions.) It is also possible to view the intervals $[\alpha/\nu,(\alpha+1)/\nu)$, rather than the points α/ν, as the atoms on which the counting measure μ operates (this is the approach taken by Anderson in [**1**]). μ-measurable functions are then the internal step-functions defined on the partition of [0, 1) determined by α/ν for $0\leqslant\alpha<\nu$, $\alpha\in N$, and, for n-integrable $f:[0, 1)\to R$, $\int f dn=\sup\{\text{standard } r\in R\,|\,r=\mathrm{st}(\int\phi d\mu)\}=\inf\{\text{standard } r\in R\,|\,r=\mathrm{st}(\int\psi d\mu)\}$ where ϕ and ψ are internal step-functions approximating $f\circ\mathrm{st}$ from below and above, respectively.

We refer the reader to [**6**] and [**1**] for further results, as well as for applications of nonstandard measure theory to the theory of probability.

7. Final remarks. I. *Logical foundations for Nonstandard Set Theory*. In this section we describe the logical foundations for Nonstandard Set Theory somewhat more rigorously. We select Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC) as a foundation for ordinary, "standard" mathematics. The only objects considered in ZFC are sets (i.e., variables $x,y,\ldots$, $A,B,\ldots$ of ZFC range over sets) and the only primitive concept is the membership relation denoted by the symbol $\in$. In addition, the language of set theory contains symbols for equality ($=$), logical connectives "and" ($\wedge$), "or" ($\vee$), "not" ($\sim$), "implies" ($\Rightarrow$), and "if and only if" ($\Leftrightarrow$), and quantifiers "for all" ($\forall$) and "there exists" ($\exists$). Formulas of ZFC are constructed from these symbols according to inductive syntactic rules:

(i) $(X\in Y)$ and $(X=Y)$ are formulas if X and Y are variables;

(ii) $(\Phi\wedge\Psi)$, $(\Phi\vee\Psi)$, $(\sim\Phi)$, $(\Phi\Rightarrow\Psi)$ and $(\Phi\Leftrightarrow\Psi)$ are formulas if Φ and Ψ are formulas;

(iii) $(\forall X)\Phi$ and $(\exists X)\Phi$ are formulas if Φ is a formula and X is a variable.

Only those statements about sets, and those descriptions of properties of sets, that are expressible by formulas, are allowed. Of course, for convenience one introduces other symbols for specific relations, constants, and operations, for example, the symbol $\subseteq$ for set inclusion. However, such symbols can be viewed as merely a shorthand for their defining formulas; e.g., $A\subseteq B$ is a shorthand for

$$(\forall x)(x\in A\Rightarrow x\in B). \tag{1}$$

One concludes the development of ZFC by specifying certain statements about sets as axioms. (Since the actual axioms are of little interest to us here, we will not state them.) All theorems about sets have to be provable from these axioms.

In Nonstandard Set Theory (NZFC) we have to deal with three sorts of objects: standard sets, internal sets, and external sets (the most encompassing sort). Variables of NZFC therefore range over the external sets; but, in addition to $\in$, we need two other primitive concepts: the relations of being standard ($\mathcal{S}$) and internal ($\mathcal{I}$). Formulas of NZFC are then constructed according to the rules (i′), (ii), and (iii), with

(i′): $(X\in Y),(X=Y),\mathcal{S}(X),\mathcal{I}(X)$ are formulas if X and Y are variables and (ii),(iii) as before.

Although any formula Φ of ZFC is also a formula of NZFC, its meaning in the two theories is, in general, different. The reason is that quantified variables range over all external sets in NZFC, and over all standard sets in ZFC. If we want to write down a formula $\Phi^{\mathcal{S}}$ of NZFC having the same meaning as a given formula Φ of ZFC, we have to *relativize* all quantifiers in Φ to $\mathcal{S}$, i.e., replace each $(\forall X)\cdots$ by $(\forall X)(\mathcal{S}(X)\Rightarrow\cdots)$ and each $(\exists X)\cdots$ by $(\exists X)(\mathcal{S}(X)\wedge\cdots)$. In full detail, the inductive rules for construction of the *standardization* $\Phi^{\mathcal{S}}$ of Φ are as follows:

(i*) $(X \in Y)^{\mathcal{S}}$ is $(X \in Y)$; $(X = Y)^{\mathcal{S}}$ is $(X = Y)$;

(ii*) $(\Phi \wedge \Psi)^{\mathcal{S}}$ is $(\Phi^{\mathcal{S}} \wedge \Psi^{\mathcal{S}})$, $(\sim\Phi)^{\mathcal{S}}$ is $(\sim\Phi^{\mathcal{S}})$, and similarly for other connectives;

(iii*) $((\forall X)\Phi)^{\mathcal{S}}$ is $(\forall X)\ (\mathcal{S}(X) \Rightarrow \Phi^{\mathcal{S}})$, more conveniently denoted $(\forall$ standard $X)\Phi^{\mathcal{S}}$, and $((\exists X)\Phi)^{\mathcal{S}}$ is $(\exists X)\ (\mathcal{S}(X) \wedge \Phi^{\mathcal{S}})$, denoted $(\exists$ standard $X)\Phi^{\mathcal{S}}$.

In an entirely analogous way we can construct, for any formula Φ of ZFC, its *internalization* $\Phi^{\mathcal{I}}$; the appropriate rules are obtained from (i*)–(iii*) by replacing $\mathcal{S}$ with $\mathcal{I}$ and "standard" with "internal."

Formulas of the form $\Phi^{\mathcal{S}}$ [$\Phi^{\mathcal{I}}$, respectively] where Φ is a formula of ZFC, are called *standard* [*internal*, respectively]. Notice that $\mathcal{I}$ [$\mathcal{S}$, respectively] does not occur in them at all, and $\mathcal{S}$ [$\mathcal{I}$, respectively] can occur only in the form specified by (iii*) (i.e., as a part of a relativized quantifier). If $\Psi = \Phi^{\mathcal{S}}$ is a standard formula, then $\Phi^{\mathcal{I}}$ is also called the internalization of Ψ, and is denoted $\Psi^{\mathcal{I}}$.

Any concept (relation, constant, operation) defined in ZFC by a formula Φ therefore splits into three, generally distinct, concepts in NZFC: the standard analog defined by $\Phi^{\mathcal{S}}$, the internal analog defined by $\Phi^{\mathcal{I}}$, and the external analog defined by Φ. For example, the relation $\subseteq$ defined by the formula from (1) has three analogs in NZFC:

$$A \subseteq^{\mathcal{S}} B \text{ defined by } (\forall \text{ standard } x)\ (x \in A \Rightarrow x \in B);$$

$$A \subseteq^{\mathcal{I}} B \text{ defined by } (\forall \text{ internal } x)\ (x \in A \Rightarrow x \in B); \text{ and}$$

$$A \subseteq^{\mathcal{E}} B \text{ defined by } (\forall x)\ (x \in A \Rightarrow x \in B).$$

For standard A and B, $\subseteq^{\mathcal{S}}$ has the same meaning as $\subseteq$ in ZFC, and we therefore write simply $\subseteq$ in place of $\subseteq^{\mathcal{S}}$. Whether or not the other two concepts, $\subseteq^{\mathcal{I}}$ and $\subseteq^{\mathcal{E}}$, coincide with $\subseteq$ is a matter requiring some analysis, as explained in §§2 and 3. When constructing $\Phi^{\mathcal{S}}$ [$\Phi^{\mathcal{I}}$, respectively], we have to replace all previously defined concepts mentioned in Φ by their standard [internal, respectively] analogs.

The axioms for NZFC then consist of standardizations of the axioms of ZFC, together with the principles introduced in §§2 and 3 (see [3] for a rigorous complete presentation).

Although we attempted to motivate our principles intuitively, that of course does not guarantee their consistency. However, we proved in [3] that *NZFC is a conservative extension of ZFC*; that is, every standard theorem which can be proved in NZFC can also be proved in ZFC alone. This means, in particular, that NZFC is consistent, assuming that the standard set theory is consistent. Moreover, the proof of this result provides a procedure which automatically translates a nonstandard proof of a standard theorem into its standard proof. In most cases, the standard proofs obtained in this manner are not very enlightening; but then one reason for using nonstandard methods lies precisely in the fact that nonstandard proofs may be more transparent and easier to find. (In [3], the theory NZFC is called $\mathfrak{NS}_2$, and the result discussed presently is stated as Theorem 2. Its proof there does not require material from the very technical §2.)

II. *Nonstandard Analysis according to Robinson*. The usual framework for Nonstandard Analysis differs somewhat from the viewpoint described in this paper. We want to make a few comments on the relationship between the two approaches in order to facilitate transition to the published literature. One conspicuous feature of the usual approach, e.g., that of Robinson [10], is the use of higher-order nonstandard models containing all entities pertinent to the particular investigation (natural numbers, reals, etc.). From our point of view, the universe of external sets plays the role of such models; it contains all entities pertinent to any investigation, namely, all standard sets. However, the main difference is that, from the usual viewpoint, the standard sets are considered as having standard elements only. Generally speaking, this makes the usual standard concepts correspond to our external concepts. We give several examples. Our standard natural numbers have only standard elements, and this makes the usual standard natural numbers correspond to our standard (i.e., external) natural numbers. However, the usual

standard set of natural numbers corresponds to our external set of natural numbers ${}^{\mathcal{E}}N = {}^{\circ}N$, rather that our N (which corresponds to what is usually denoted by N^*). Similarly, the usual standard set of rationals corresponds to our ${}^{\circ}Q$. The usual standard set of reals corresponds to our external set of external reals, ${}^{\mathcal{E}}R$.

More formally, we can define *external standard sets* and a mapping * inductively as follows: $\mathbf{A}$ is an external standard set, $\mathcal{S}^{\mathcal{E}}(\mathbf{A})$, if all elements of $\mathbf{A}$ are external standard sets, and then $\mathbf{A}^* = {}^*\{\mathbf{a}^* | \mathbf{a} \in \mathbf{A}\}$. Note that the induction starts with the empty set. For each external standard $\mathbf{A}$, $\mathbf{A}^*$ is a standard set; for example, $({}^{\mathcal{E}}N)^* = N$, $({}^{\mathcal{E}}Q)^* = Q$, $({}^{\mathcal{E}}R)^* = R$. (Note also that this mapping * is not the same as the one defined in §3.) It is easy to prove inductively (and using the Principle of Standardization) that * is an isomorphism, with respect to $\in$, between the universe of external standard sets and the universe of standard sets. It is thus possible to adopt the usual viewpoint and consider the universe of (external) standard sets as embedded into the universe of internal sets by *. The Principle of Embedding then says that, for any set-theoretic property Φ, and any (external) standard $\mathbf{A}, \mathbf{B}, \ldots$, $\Phi^{\mathcal{S}^{\mathcal{E}}}(\mathbf{A}, \mathbf{B}, \ldots)$ if and only if $\Phi^{\mathcal{I}}(\mathbf{A}^*, \mathbf{B}^*, \ldots)$, i.e., * is an elementary embedding of the universe of (external) standard sets into the universe of internal sets. Similarly, the Principle of Saturation can be reformulated from this viewpoint. With a little practice, the reader should have no difficulties using either approach.

III. *Other Axiomatic Systems for Nonstandard Analysis.* My work on an axiomatic approach to Nonstandard Analysis began in the spring of 1972, when, stimulated by interest taken in the subject by students in my model theory course, I formulated the axioms of NZFC ($\mathfrak{NS}_2$) and proved that it is a conservative extension of ZFC. An analogous result for a related system ($\mathfrak{NS}_1$) turned out to be much harder, and the paper [3] was completed only in late 1974 (see also an abstract [2]). Since then, I have learned about two other attempts to develop nonstandard mathematics axiomatically.

Starting in 1973, *P. Vopěnka* and his students devised an *Alternative Set Theory* and developed some areas of mathematics (in particular, calculus and general topology) in this framework. We refer the interested reader to [13] for a detailed exposition and other references. It might be helpful to note that, by taking all external subsets of the standard set HF of all hereditarily finite sets, one obtains a model of the Alternative Set Theory (except for one axiom, essentially the Continuum Hypothesis).

E. Nelson developed an axiomatic theory called the *Internal Set Theory* in [9], and showed how a variety of nonstandard constructions can be performed in it. The Internal Set Theory deals only with standard and internal sets; its axioms correspond roughly to our principles of Embedding, Weak Saturation (there are differences here: saturation with respect to a proper class of standard parameters is allowed, the property Φ may have internal parameters (as in our Strong Saturation), and the principle is stated as an equivalence, rather than an implication) and Standardization (for external sets definable in the language of the Internal Set Theory). External sets can be introduced for the price of working with a model of set theory; this is satisfactory when external sets are used for bookkeeping purposes only, but becomes less wieldy when an interplay between external and internal concepts is important, such as in constructions of §6. (There, for example, a countable model of set theory would not do, because the external reals could not be identified with the standard ones.) The paper [9] contains many detailed explanations of pitfalls in nonstandard reasoning, and would be good reading for those interested in learning more about the developing field of Nonstandard Mathematics.

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FIFTY YEARS AGO

A fairly random selection from the topics of the 46 articles in volume 36 of this MONTHLY includes: "How can interest in calculus be increased?"; "The fundamental mathematical requirements of biology" (by a botanist); differential equations in electrical circuit theory, by T. C. Fry, of Bell Telephone Laboratories; Cantor's singular function, by Hille and Tamarkin; trigonometry in hyperspace; kinematics; polar conics and osculant conics of a nodal conic; the motion of a satellite of a spheroidal planet.

The winter meeting was in New York in December 1928; the summer meeting, at Boulder ("first time in the Far West"). At the former, E. R. Hedrick called for, among other things, the establishment of a mathematics abstracting journal. At the latter, there were addresses on the undergraduate mathematics curriculum in a liberal arts college and on factorization of large numbers (up to 1.5×10^9). The College Entrance Examination Board was being urged to modify its requirements so as to get more solid geometry into the high school curriculum.

In short, the character of the MONTHLY and the concerns of the Association seem to have changed rather little in half a century, except that algebra has largely replaced geometry and the Problems are now drawn from more different fields. There is little evidence of any awareness in 1929 that major changes were soon to occur in Mathematics; for example, there were no articles on topology or abstract algebra or (what was to become) functional analysis. The problems of teaching appear to have remained much the same for fifty years, but I hope that the MONTHLY may now be bringing its readers more awareness of how Mathematics itself is changing.

Volume 36 contained 560 pages (in larger type and with wider margins than the one you are reading) and a 67-page Register of officers and members. There were 1,964 individual members, whose dues were $4 a year. The winter meeting was attended by 340 people; the summer meeting, by 239. In 1928, American universities awarded 49 doctorates in Mathematics (they included A. A. Albert, L. W. Cohen, B. W. Jones, Morris Marden and Morgan Ward). Fine Hall, at Princeton, was about to be built at a cost of $400,000.

Notice the curious effects of various kinds of inflation: in 1979 we have about 10 times as many members, each paying more than 5 times as high dues, but the MONTHLY is only slightly more than twice as large and actually publishes fewer (although longer) main articles than in 1929. The number of short notes has approximately doubled; the number of extended book