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MATHEMATICAL FOUNDATIONS FOR MATHEMATICS

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1. Introduction. Most mathematical papers deal with mathematics "in the small"—a few definitions, a few theorems, a few proofs. If the author has a modicum of boldness and compassion he may also include some account of the intuitive ideas from which these formal parts of his work were fashioned. This paper, however, will have a different character.

In wondering what subject to choose for this Chauvenet Symposium, I let my mind's eye wander over those areas of the foundations of mathematics in which I have worked or dabbled—completeness proofs, applications of logic to algebra, decision problems, infinitary logic, algebraic logic . . . somehow none of them seemed appropriate. I began to wonder why. Presently it seemed to me that the answer was bound up with what might be called the "sociological structure" of our contemporary American mathematical community.

[1] The point of view toward foundations developed here was first formulated by me in the IBM Lectures which I gave at Swarthmore in December, 1967. This viewpoint has been developed over an extended period, during which much of my work was supported by the National Science Foundation (most recently, Grant No. GP-6232X).

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Professor Leon Henkin received his PhD at Princeton University in 1947 under the direction of Alonzo Church. His thesis included a proof of Gödel's completeness theorem for the predicate calculus which has since become the standard proof in almost every presentation of mathematical logic. In addition Professor Henkin developed the theory of cylindrification algebras which is an algebraic formulation of the theory of quantifiers. His principal work has been in the area of foundations and mathematical logic in which he has published many papers and is a recognized authority. He was awarded the Chauvenet Prize in 1964 for his paper "Are Logic and Mathematics Identical?" published in Science, 1962 (vol 138).

Professor Henkin was Fine Instructor and Jewett Fellow at Princeton from 1947 to 1949, having spent four previous years as a mathematician in industry. In addition he was a Fullbright Scholar in Amsterdam in 1954–55, a Visiting Professor at Dartmouth in 1960–61, and a Guggenheim Fellow and member of the Institute For Advanced Study in 1961–62, and a visiting Fellow at All Souls' College, Oxford in 1968–69. He taught at the University of Southern California and has been a member of the faculty of the University of California at Berkeley since 1953 where he has served as chairman twice. He has been Editor for the *Journal of Symbolic Logic* and served a three year term as President for the Association of Symbolic Logic. He has also been a member of the Council of the American Mathematical Society as well as active in CUPM. Besides his papers in foundations he is the author of "Retracing Elementary Mathematics," Macmillan, 1962, which indicates his keen interest in the teaching of mathematics. *J. C. Abbott.*

For the fact is that mathematicians who work in areas of foundations are considered by most other mathematicians to be somewhat "different." Those working in any of the central parts of algebra, geometry, or analysis are generally familiar with the most basic notions in each other's domains, and have at some point worked on elementary problems dealing with these notions. But foundations, along with other "new" mathematical areas such as statistics or computer science, and certain "old" areas such as celestial mechanics, is considered a domain of whose details most mathematicians may safely remain ignorant, as long as they know in a vague way what the subject is about. These "fringe" areas of mathematics each relate mathematics to some non-mathematical discipline; but while, in the case of statistics, computer science, or celestial mechanics, this outside discipline is accepted as a proper domain for the application of mathematics; in the case of foundations the outside discipline is philosophy—and this renders the area just a little bit suspect.

I believe that this state of affairs makes it difficult for a "foundationist" to write an ordinary "in the small" paper for a general mathematical audience, unless it deals with the most elementary concepts. I have decided, therefore, to try to write about the foundations of mathematics "in the large."

Actually the phrase "foundations of mathematics" has a meaning which seems to shift from one context to another. Sometimes it denotes a part of mathematics—a sort of beginning portion on which all the other parts are built. At other times this phrase refers to a commentary on mathematics from the outside—a sort of explanation of the significance of the work of mathematicians, couched often in metaphysical terms.

If we take the pragmatic viewpoint that the foundations of mathematics is that which is done by those who profess to work in this area, two things become apparent. First, the *extent* of the domain, the number and distance-from-one-another of the subjects pursued under the heading of foundations of mathematics, has increased enormously during the course of recent decades. And second, the center of mass of this profusion of material has been moving continuously, and at an accelerated rate, from philosophy into mathematics.

Such a transition is by no means unique to the foundations of mathematics. All the empirical sciences, in their time, have been located squarely within "natural philosophy"; only gradually did they detach themselves and become separate disciplines. The independent fashioning of such sciences as physics, chemistry, and biology has not meant that philosophers have lost all interest in these disciplines. Quite the contrary, philosophers remain interested spectators, and to some extent even participants, in their ongoing development. And so it is with the foundations of mathematics. Even though the mathematician is increasingly taking up the subject and transmuting it thereby into a corpus having the form of mathematical definitions, theorems, and proofs, there remains nevertheless a residue of inquiry beyond the borders of mathematics which is a fertile—and by no means neglected—area for philosophical inquiry.

The entrance of mathematicians as a significant and recognized component of those working in the field of foundations is generally dated around the turn of the century. The ground had been prepared by the development of new parts of mathematics, beginning with non-Euclidean geometry in the first half of the 1800's, Boole's work on logic about 1850, Cantor's creation of a theory of sets from 1870 on, and including the work of such men as Dedekind, Schroeder, Frege, Weierstrass, Peano, and Zermelo. But it remained for other mathematicians, writing somewhat later, to delineate and urge *viewpoints* concerning the foundations of mathematics, based upon these and further technical developments, which exerted intense and lasting influences on the further development of the subject. Best known among these were Kronecker, Poincaré, Brouwer, Hilbert, and Russell.

These men whose work in various areas of mathematics had in most cases already won them wide renown, engaged in a public debate concerning the nature and significance of mathematics, through a series of publications, and through letters which subsequently became public. Such fundamental controversy, almost unknown in the older branches of mathematics, naturally became the object of broad attention, and out of it there emerged three "schools" which have come to be known under the titles Intuitionism, Formalism, and Logicism.

Intuitionism, as Brouwer developed it from tendencies put forward by Kronecker and Poincaré, became a radical renunciation of the classical use of mathematical language and logic, and a demand for a severe limitation on the permissible kinds of mathematical constructions. Formalism, as Hilbert's program came to be known, focused attention sharply on the symbolic patterns of mathematical language, and attempted to explain their use in terms of rules having a syntactical character, whose ultimate justification was to be achieved by consistency proofs of so fundamental a character as to put them beyond dispute. Logicism, as Russell based its exposition on Frege's ideas, concentrated on the unification of mathematics, through systematic reduction of its parts to the most elementary and the most general—logic and the notion of sets.

In the six or seven decades since these positions were so ably set forth, mathematicians have taken up foundational work in steadily increasing volume. Much of the early material stemmed from, and fitted neatly into, the basic tripartite framework outlined above. But as in any lengthy mathematical development, continuous transformations, as well as abrupt changes resulting from completely unexpected developments, reshaped the discipline. New areas of foundational work appeared and new connections were found between foundations and other parts of mathematics.

Despite this burgeoning of mathematical activity, *commentators* who seek to classify and analyze the work have continued for the most part to try to fit the details into the logicism-formalism-intuitionism scheme. To us it appears that these efforts are increasingly unavailing, and lead to a more and more distorted view of what is being done. Perhaps the most basic reason is that the concepts embodied in these viewpoints are essentially philosophical in character, and as such are not well adapted for describing material in accelerated transition from philosophy into mathematics.

Thus the time is at hand, it seems to us, to put forward a new framework, or classification scheme, within which cohesion, form, and meaning can be given to the totality of work in the foundations of mathematics. The concepts underlying such a classification scheme should be essentially mathematical in character, rather than philosophical, to take proper account of the increasingly mathematical nature of the work to be classified. The particular suggestion we shall advance here is that the continuing analysis of foundational work be based on the following classification scheme: set-theoretical aspects, algebraic aspects, and constructive aspects.

It is, of course, possible to consider that such a scheme is little more than a renaming of the same basic categories we have described earlier. Certainly the notions of set theory form a basic part of logicism, algebraic aspects of foundations can be closely related to the formalist's approach, and constructive elements of mathematics often arise from the same spirit of inquiry which motivated Brouwer to hoist the pennant of intuitionism. But though it is worthwhile to notice these connections, it seems to me that our proposed classification scheme differs from what has become the traditional one, in some fundamental ways.

One of the most basic differences between these two schemes has to do with the question of compatibility. Logicism, formalism, and intuitionism are essentially *competitive* ways of viewing the foundations of mathematics. Indeed, the original exchanges of publications and letters by the authors of these programs bordered on the acrimonious, because each considered that his was the *right* way to look at the foundations of mathematics, and that the others were (in consequence) *wrong*. By contrast, we find today a considerable cooperation among those pursuing set-theoretical, algebraic, or constructive elements in the foundations of mathematics. Each of these strands of the subject supplements the others; they are interwoven to provide a richly illuminated depiction of a common domain. Some of the most prized theorems and valuable insights are those which illuminate the interconnections among these strands of foundational work.

In the remainder of this paper we wish to illustrate the working of our proposed conceptual framework by indicating how some of the major complexes of problems which have occupied "foundationists" can be viewed within it.

2. Algebraic aspects of foundations. I still recall the definition of algebra given by my high school teacher when I took my first course in the subject: "It is like arithmetic," she said, "but we work with letters instead of numbers." Those who have pursued the subject beyond high school will be unable to give so simple an account of this part of mathematics, while agreeing that the use of letters as variables in equations remains an important characteristic of the subject.

One of the remarkable trends of recent mathematics has been the development of algebraic parts of various disciplines—we have algebraic topology, for example, algebraic geometry, functional analysis, and most recently algebraic logic. This incursion of algebra into other parts of mathematics represents, I believe, an important part of the effort to preserve a unity in mathematics, so that the development of more specialized disciplines may be counterbalanced by mathematical elements which facilitate communication among *all* mathematicians.

Although the term "algebraic logic" (coined by Halmos) is less than ten years old, the ideas of the subject are easily traced back along a continuous path to Boole's work in 1850. Indeed Boole's starting point was the observation that certain classical laws of logic could be expressed by means of algebraic equations; for example, the logical equivalence of a proposition with the denial of its denial took the form $--p=p$, while the law of the excluded middle was expressed by the identity $p+-p=1$. The fact that one could pass from such laws to others by the familiar algebraic methods of substitution for the variables of an equational identity, and the "replacement of equals by equals," served further to show an underlying identity between logic and the algebra of numbers.

Boole further enriched his theory by providing a second interpretation for his algebraic equations, in which variables represented sets instead of propositions. The equation $--p=p$ then came to mean that the complement of the complement of a set was the set itself, while $p+-p=1$ expressed the fact that the union of a set and its complement was the universal set. It remained for Stone, however, some eighty years later, to carry the interpretation of Boole's equations to the ultimate possibility by considering the class of *all* structures which satisfy them, calling these Boolean algebras, and studying their relations to one another in terms of such concepts as homomorphisms, subalgebras, and direct products which had evolved in the study of rings and groups.

Between Boole and Stone, however, there was other activity. Boole's calculus of classes led to an exhaustive study of a calculus of binary relations by Schroeder; Peirce took important steps toward bringing quantifiers within the scope of algebraically-expressed logic; and Tarski initiated a study of deductively closed systems of sentences which anticipated several of Stone's findings and methods.

The use of Boole's equations to define the class of Boolean algebras, and the central role played in their theory by Stone's representation theorem, had a profound effect on the further development of algebraic logic. One of the first of these was the establishment of a theory of relation algebras by Tarski: he selected some of the equational identities discovered by Schroeder in his study of binary relations, and used them as axioms to define a new class of algebras. He then studied the possibility of isomorphically representing arbitrary algebras of this class by means of binary relations over some set. This inquiry had an unusual history, as we shall relate below.

In studying the set of all binary relations over some set U one distinguishes not only the empty relation, $\varnothing$, and the universal relation $U\times U$, but also the identity relation I_U. And one studies not only the Boolean operations $\cup$, $\cap$, and $\sim$ (complementation with respect to $U\times U$, under which the set of binary relations on U is closed), but also the operation of forming the converse $R^{\smile}$ for any given $R\subseteq U\times U$, and the binary operation of forming the relative product $R|S$ for given R and S, where for any x, $y\in U$ we have $x(R|S)y$ iff xRz and zSy for some $z\in U$. The Boolean identities form a trivial part of the theorems of this theory. Many of the identities involving I_u, $^{\smile}$, and $|$, are also simple: For example the associative law for $|$, the fact that I_u is a two-sided identity element for the operation $|$, the distributive law for $^{\smile}$ over $\cup$, the fact that $|$ is additive (i.e., that $(R_1\cup R_2)|S=(R_1|S)\cup(R_2|S)$ and $R|(S_1\cup S_2)=(R|S_1)\cup(R|S_2)$), and the law $(R|S)^{\smile}=S^{\smile}|R^{\smile}$. On the other hand, there are many equations involving these notions for which it is very difficult to determine whether they hold identically for relations over an arbitrary set U. Indeed, Tarski has shown that there does not exist a decision method which can determine automatically, in a finite number of steps, whether any given equation is such an identity. He does this by showing how an arbitrary statement of elementary number theory (a theory known to be undecidable) can be translated into an equivalent statement having the form of an equational identity for relation algebras. By contrast, it is known that a Boolean equation holds identically in any Boolean algebra iff it holds identically for the algebra of subsets of a one-element set, which of course provides a simple decision method for selecting the Boolean identities from among all Boolean equations.

After Tarski had selected certain identities from the theory of relations and used them as axioms to define the class of relation algebras, other identities were found, by Lyndon, which could not be derived from these axioms. It was natural to think of adding these new identities as further axioms, in an effort to obtain a system of axioms sufficiently strong to imply all identities of the theory of binary relations or, better still, strong enough to allow the inference that any structure satisfying the axioms must be isomorphic to an algebra of relations over some set. However, Lyndon stated that the latter goal was unattainable by presenting a relation algebra not isomorphic to any algebra of binary relations over a set U, which seemed to satisfy all the same equational identities as a certain algebra of the latter kind. Later, however, Tarski demonstrated a contrary result: He showed that any model satisfying the set of all equational identities which hold in every algebra of relations, must be isomorphic to such an algebra. (This he did by employing Garrett Birkhoff's criterion for equational classes of structures, showing that the class of isomorphic images of algebras of relations is closed under formation of subalgebras, homomorphic images, and direct products.) The mistake in Lyndon's proof was later found by Dana Scott. Still later, Monk showed that no finite system of identities can characterize the isomorphic images of algebras of relations.

A different approach to the theory of binary relations was taken by Everett and Ulam, who formulated the notion of a projective algebra. In place of the operations of converse and relative product, they consider the projections P_0 and P_1 of an arbitrary relation $R \subseteq U \times U$ onto the "lines" $\{e\} \times U$ and $U \times \{e\}$, where e is a distinguished element of U; the Boolean operations on relations are retained in these structures. This approach algebraizes a much more limited part of the theory of relations than Tarski's, and the authors are able to establish a representation theorem for the class of projective algebras. It has also been shown that there is a decision procedure to determine automatically, in a finite number of steps, whether a given equation in this theory holds identically for arbitrary binary relations (or—equivalently, in view of the representation theorem—for arbitrary elements in any projective algebra).

Although the notion of projective algebra led to a very restricted theory, it suggested to Tarski, and his students Chin and Thompson, certain modifications which have led to a much richer development. The most significant change was to abandon a restriction to binary relations over a set U, and to consider relations of rank α over U for arbitrary α. Next, instead of considering the projections of a relation on some special "line" (or linear subspace of higher dimension for relations of higher rank), the fundamental operations are chosen to be the cylindrifications C_κ for each $\kappa < \alpha$: For any relation R of rank α over U, the relation $C_\kappa R$ is the set of all "points" $(x_0, \cdots, x_{\alpha-1})$ of ${}^\alpha U$ such that

$$(x_0, \cdots, x_{\kappa-1}, y, x_{\kappa+1}, \cdots, x_{\alpha-1}) \in R$$

for some $y \in U$. Finally, in case $\alpha = 2$ the identity relation over U is taken as a distinguished element just as in the case of relation algebras and, more generally, for any α we distinguish the "diagonal relations" $D_{\kappa\lambda}$ of rank α over U, where $(x_0, \cdots, x_{\alpha-1}) \in D_{\kappa\lambda}$ iff $x_\kappa = x_\lambda$ (for each $\kappa, \lambda < \alpha$). A set of relations of rank α over U is called a *cylindric field of dimension* α if it contains the relations $\emptyset$, ${}^\alpha U$, and each $D_{\kappa\lambda}$ (for $\kappa, \lambda < \alpha$), and if it is closed under the Boolean operations $\sim$ (complementation with respect to ${}^\alpha U$), $\cup$, $\cap$, and all of the cylindrifications C_κ (for $\kappa < \alpha$). The notion of a *cylindric algebra of dimension* α is obtained by abstracting from the notion of a cylindric field of relations; such structures have the form

$$\langle A, +, \cdot, -, 0, 1, d_{\kappa\lambda}, c_\kappa \rangle_{\kappa,\lambda<\alpha}$$

where $\langle A, +, \cdot, -, 0, 1 \rangle$ is an arbitrary Boolean algebra, $d_{\kappa\lambda} \in A$ and c_κ is a one-place operation on A for each $\kappa, \lambda < \alpha$, and in which certain equations are satisfied identically. These equations, collected into seven axiom schemata, are of course chosen from among those equations known to hold identically in every cylindric field. Among these we may cite:

AXIOM C_3. *$c_\kappa(x \cdot c_\kappa y) = c_\kappa x \cdot c_\kappa y$ for all $x, y \in A$.*

AXIOM C_7. *$c_\kappa(d_{\kappa\lambda} \cdot x) \cdot c_\kappa(d_{\kappa\lambda} \cdot -x) = 0$ for all $x \in A$, where $\kappa, \lambda < \alpha$ and $\kappa \neq \lambda$.*

The feature that gives to cylindric algebras a special importance for logical studies is the fact that the cylindrifications c_κ stand in exactly the same relation to the existential quantifiers $(\exists v_\kappa)$ of predicate logic, as the Boolean operations

$+, \cdot, -$ bear to the connectives $\vee$ (disjunction), $\wedge$ (conjunction), and $\neg$, (negation) of sentential logic. Also the diagonal elements $d_{\kappa\lambda}$ correspond algebraically to the equations $v_\kappa = v_\lambda$ of predicate logic with equality. Thus the study of these algebras provides a way to deal with questions of predicate logic by algebraic methods. However, before we can be satisfied that the algebraic structure is adequate for a full logical analysis there are certain questions which must be considered.

In predicate logic, reflecting the needs of many parts of mathematics, we generally consider systems in which several relations of different ranks may be considered. How do we deal with these in the context of a cylindric algebra of dimension α, where we have one fixed value of α? Very simply, we choose α to be ω (the first transfinite ordinal), and for each integer κ we represent a relation of rank κ as a relation of rank ω which depends on only the first κ coordinates. For instance, if we take U to be the set of all real numbers and we wish to represent the binary relation $<$ on U as a relation of rank ω, we would use the relation R such that, for every infinite sequence $(x_0, x_1, \cdots)$ of real numbers we have $(x_0, x_1, \cdots) \in R$ iff $x_0 < x_1$.

In this example, we notice that if $\kappa \geqq 2$ and $(x_0, x_1, \cdots, x_\kappa, \cdots) \in R$ for some sequence $(x_0, x_1, \cdots, x_\kappa, \cdots)$ of real numbers, then if we change the coordinate x_κ in any way the resulting sequence is also in R; in other words we have $C_\kappa R = R$. More generally, whenever we represent a relation of rank λ as a relation R of rank ω, we shall have $C_\kappa R = R$ for every $\kappa \geqq \lambda$. Thus, in an arbitrary cylindric algebra A of dimension ω, if we consider the set A_λ of all those $x \in A$ for which $c_\kappa x = x$, whenever $\kappa \geqq \lambda$, we shall be dealing with an algebraic version of a cylindric field of relations of rank λ; it is not surprising, therefore, that the set A_λ together with the operations $c_0, \cdots, c_{\lambda-1}$ and the diagonal elements with indices $<\lambda$ forms a cylindric algebra of dimension λ.

Of course there are relations of rank ω which do not depend on only a finite number of coordinates, and hence which do not correspond to any relation of finite rank. In ordinary predicate logic, and most parts of mathematics, we deal only with relations of finite rank. Hence, if we wish to consider cylindric algebras which correspond precisely to systems of predicate logic, we impose the condition that for every $x \in A$ there is some $\lambda < \omega$ such that $c_\kappa x = x$ for all $\kappa \geqq \lambda$. Such a cylindric algebra is said to be *locally finite*.

Finally, let us consider the operation of forming the converse of a binary relation, one of the fundamental notions of relation algebras. In predicate logic the corresponding operation is a substitution, enabling us to pass from a formula Gxy to the formula Gyx. How can we obtain a corresponding operation in the theory of cylindric algebras? To see this, consider first the operation S_2^0 on relations, such that $S_2^0 R = C_0(D_{02} \cap R)$ for each R. If R is a relation of rank ω which represents a binary relation, so that $C_\kappa R = R$ for all $\kappa \geqq 2$, then we shall have

$$(x_0, x_1, x_2, x_3, \cdots) \in S_2^0 R \qquad \text{iff } (x_2, x_1, x_0, x_3, \cdots) \in R.$$

For the relation thus obtained we have $C_\kappa(S_2^0R)=S_2^0R$ for $\kappa=0, 3, 4, 5, \cdots$. Next, applying the operation S_3^1 to S_2^0R (where, in general, $S_3^1T=C_1(D_{13}\cap T)$ for any relation T), we find that

$$\begin{aligned}(x_0, x_1, x_2, x_3, \cdots) \in (S_3^1S_2^0R) \quad &\text{iff } (x_0, x_3, x_2, x_1, \cdots) \in (S_2^0R)\\ &\text{iff } (x_2, x_3, x_0, x_1, \cdots) \in R,\end{aligned}$$

and $C_\kappa(S_3^1S_2^0R)=S_3^1S_2^0R$ for $\kappa=0, 1, 4, 5, 6, \cdots$. Continuing in this way by applying successively the operations S_0^3 and S_1^2 we finally obtain the relation $T=S_1^2S_0^3S_3^1S_2^0R$ such that $C_\kappa T=T$ for each $\kappa\geqq 2$ and

$$(x_0, x_1, x_2, x_3, \cdots) \in T \qquad \text{iff } (x_1, x_0, x_2, x_3, \cdots) \in R.$$

Clearly, then, T is a relation of rank ω representing the converse of the binary relation represented by R.

Similarly, in an arbitrary cylindric algebra of dimension α we can define an operation S_λ^κ for each κ, $\lambda<\alpha$ so that $S_\lambda^\kappa x=C_\kappa(d_{\kappa\lambda}\cdot x)$ if $\kappa\neq\lambda$ and $S_\kappa^\kappa x=x$. If $\alpha=\omega$ and the algebra is locally finite, we can find a suitable finite sequence of these operations S_λ^κ to effect any given substitution on relations of given rank. For example, suppose S' is the substitution on relations such that

$$(x_0, x_1, x_2) \in S'R \qquad \text{iff } (x_2, x_0, x_0) \in R$$

for any relation R of rank 3. Then S' corresponds to the operation $S_0^3 \circ S_0^1 \circ S_2^0 \circ S_3^2$, in the sense that for any relation R of rank ω such that $C_\kappa R=R$ for all $\kappa\geqq 3$, we shall have also $C_\kappa(S_0^3S_0^1S_2^0S_3^2R)=(S_0^3S_0^1S_2^0S_3^2R)$ for all $\kappa\geqq 3$ and

$$(x_0, x_1, x_2, x_3, \cdots) \in (S_0^3S_1^0S_2^0S_3^2R) \qquad \text{iff } (x_2, x_0, x_0, x_3, \cdots) \in R$$

for each sequence $(x_0, x_1, x_2, x_3, \cdots)$.

We have thus seen how a locally finite cylindric algebra of dimension ω is equipped to deal with relations of any finite rank λ, and how its fundamental operations of cylindrification can be combined with its diagonal elements to permit the definition of an arbitrary operation of substitution on relations of given rank λ. Thus all of the elements to be found in a system of first-order predicate logic have their algebraic counterparts in cylindric algebras of this type.

For the class of locally finite cylindric algebras of dimension ω, there is the following representation theorem: Given any such algebra A, and any of its elements $x\neq 0$, there exists a homomorphism h of A into a suitable cylindric field relation, such that $hx\neq 0$. (By considerations of a general algebraic character, one can show that this is equivalent to the statement that every algebra A of the class considered is isomorphic to a subdirect product of cylindric fields.) This representation theorem may be considered as an algebraic analogue of the completeness theorem for predicate logic. Indeed, the two theorems are closely related, each of them being directly deducible from the other.

If we consider the class of all cylindric algebras of dimension ω, abandoning the condition of local finiteness, the representation theorem no longer holds. The class of arbitrary cylindric algebras of dimension ω includes cylindric fields of relations of rank ω which do not represent relations of any finite rank λ, i.e., which do not satisfy any condition of the form $C_\kappa R = R$ for all $\kappa \geqq \lambda$. Such algebras correspond to infinitary systems of predicate logic in which atomic formulas may consist of a predicate symbol followed by an infinite sequence of individual symbols. The failure of the representation theorem for the class of all ω-dimensional cylindric algebras means that there is such an algebra A which is not isomorphic to any subdirect product of cylindric fields; this is shown by producing an equation which holds identically in every cylindric field of dimension ω, but fails in A. In consequence we can find, in a system of infinitary predicate logic such as described above, a set of formulas which is consistent under the ordinary rules of deduction but cannot be satisfied in any model.

Of course we can consider the class of all cylindric algebras of *any* given dimension α, finite or infinite. Also, for any α we can consider cylindric fields whose elements are relations of rank α over some set U; and so we can ask whether all α-dimensional cylindric algebras are representable. It turns out that in each case the answer is negative, except for $\alpha = 0$ and $\alpha = 1$. In the case $\alpha = 2$, we can produce two equational identities which characterize the representable 2-dimensional cylindric algebras; for any $\alpha > 3$, no finite set of equational identities characterizes the set of representable α-dimensional cylindric algebras, although an α-dimensional cylindric algebra satisfying all equational identities which hold in every cylindric field of dimension α *is* always representable.

The class of locally finite algebras has a generalization: For any infinite α, we may consider those α-dimensional cylindric algebras A such that, for every $x \in A$, we have $c_\kappa x = x$ for at least one $\kappa < \alpha$. Such an algebra is said to be *dimension-complemented*, and every such algebra turns out to be representable. Here are some other classes of cylindric algebras known to be representable for any dimension α: (i) Every algebra in which the unit element can be expressed as a finite sum of diagonal elements $d_{\kappa\lambda}$ $(\kappa \neq \lambda)$; (ii) every algebra which can be generated by monadic elements, i.e., by elements x such that $c_\kappa x = x$ for all $\kappa > 0$; (iii) every algebra A of dimension α which can be isomorphically imbedded in an algebra B of dimension $\alpha + \omega$, in such a way that for every $x \in A$ we have $c_\kappa x = x$ for $\kappa = \alpha, \alpha+1, \alpha+2, \cdots$.

There are a great many open questions in the theory of cylindric algebras. Some of these are purely algebraic in character; for example, the question whether every such algebra which is finitely generated and simple (admits only trivial homomorphisms) can be generated by a single element: this is known to have a negative answer for cylindric algebras of finite dimensions, but is open for the infinite dimensional case. Other open problems relate arbitrary cylindric algebras with cylindric fields of relations. Of these, two problems occupy a central position: (i) To find, for arbitrary α, an intrinsic, algebraic characterization of the class of all α-dimensional cylindric fields of relations; (ii) to find set-theoretic operations with intuitive geometric content, which can be performed on α-dimensional cylindric fields to yield a class of relational structures to provide an isomorphic representation of every cylindric algebra of dimension α. These questions have foundational significance because they are algebraic formulations of problems about a broad class of logical systems and their interpretations.

To complete this sketch of algebraic logic we mention that the concept of cylindric algebras is only one among several classes of algebraic structures that have been introduced in connection with logical systems. Certain classes are obtained by modifying the Boolean laws in order to deal with "nonclassical" logics. Among types of algebras employed for studying classical logic, polyadic algebras are those which, along with cylindric algebras, have been studied most intensively. Polyadic algebras are Boolean algebras whose fundamental structure is enriched by cylindrifications and by arbitrary substitutions (corresponding to the permutation and identification of individual symbols in predicate logic). The class of locally finite, ω-dimensional, polyadic algebras provides an algebraic theory of predicate logic without equality; algebras of this class which contain a set of diagonal elements are equivalent to locally-finite ω-dimensional cylindric algebras. For finite α, or for infinite α without the condition of local finiteness, the theories of these two classes have essential differences. In particular, every infinite-dimensional polyadic algebra can be represented by polyadic fields of relations.

We have been writing in some detail about the development of algebraic logic, for that is the most overt way in which algebraic ideas have entered into work on the foundations of mathematics. However, a total inventory of the algebraic aspects of foundational research would include many, many other areas. We shall mention three of these.

1. The whole development of deductive logic may be viewed as a process of algebraization. In order to understand this, let us sketch the common elements of the formation of almost any algebraic theory—theories of groups, say, or of rings, or of vector spaces. (i) We start with some particular domain of elements and operations on it. (ii) We develop a symbolic language to talk about this structure and we find sentences of the language, true of the structure, which seem to be related to one another by simple formal rules of transformation—for example, we may concentrate on sentences having the form of equational identities, related to one another by rules of substitution and replacement. (iii) We abstract from the particular domain in which we were first interested by selecting certain of the true sentences encountered in (ii) and using them as axioms, and we then study the class of all structures which satisfy those axioms. This is the heart of the process of algebraization. (iv) We look for representation theorems which relate arbitrary models of our axioms to the particular structure which gave rise to the theory. . . . This four-part process of development can be discerned in any well-developed algebraic theory.

Now how can we view deductive logic from this point of view?

(i) The most basic object of study in logic is the relation of implication. This is a relation which connects a given sentence (the conclusion) with a given set of sentences (the assumptions). Of course underlying this relation is a grammatical structure $\mathfrak{L}$ within which the sentences are constructed—it consists of a list of symbols (classified into different kinds such as variables, connectives, parentheses, etc.) and rules for combining them into categories such as terms, formulas, and sentences. Once the grammar $\mathfrak{L}$ is set forth in precise terms, it is necessary to describe how its components are *interpreted* in order to obtain a discourse language for some given structure (or other domain of discourse); under such an interpretation each sentence of $\mathfrak{L}$ takes on a definite truth value, truth or falsity. Finally, a sentence is called *logically valid* if it is true under *every* interpretation; a sentence ϕ is a *logical consequence* of a set Γ of sentences, and Γ is said to *imply* ϕ, if ϕ is true in *all those* interpretations which make every sentence of Γ true.

(ii) To talk about the set of valid sentences and the relation of implication, we introduce variables like "ϕ" and "ψ" to range over sentences of $\mathfrak{L}$, and variables like "Γ" and "Δ" to range over sets of sentences, and we introduce a special symbol "$\models$". We write $\models\phi$ to indicate that ϕ is valid; we write $\Gamma\models\phi$ to indicate that Γ implies ϕ. (This double use of the same symbol is reasonable since $\models\phi$ iff $\varnothing\models\phi$.) We also have symbols for the operations of building complex sentences from simpler ones, or from formulas, such as the familiar notations $\phi\rightarrow\psi$ for an "if . . . then" sentence or "$\forall v\phi$" denoting a "for all . . . " sentence. Finally, we find simple transformation rules allowing us to pass from certain expressions of the form "$\Gamma\models\phi$" which are true about implication, to others. For instance, the rule of detachment tells us that whenever we have $\Gamma\models\phi$ and $\Gamma\models\phi\rightarrow\psi$, we shall also have $\Gamma\models\psi$; another rule allows us to pass from $\Gamma\models\forall x(\phi\rightarrow\psi)$ to $\Gamma\models(\forall x\phi)\rightarrow(\forall x\psi)$.

(iii) Instead of the particular relation $\models$ we now abstract and consider an *arbitrary* relation, $\vdash$, satisfying certain of the conditions which we found to be satisfied for the particular relation $\models$. For example, if we have noticed that $\models\phi\rightarrow(\psi\rightarrow\phi)$ for all sentences ϕ, ψ of $\mathfrak{L}$, we may impose the axiom $\vdash\phi\rightarrow(\psi\rightarrow\phi)$ for all ϕ, ψ of $\mathfrak{L}$ on our undefined relation $\vdash$. Similarly, we may adopt the rule of detachment as an axiom on $\vdash$, requiring that whenever $\Gamma\vdash\phi$ and $\Gamma\vdash\phi\rightarrow\psi$, we also have $\Gamma\vdash\psi$ . . . . The passage from the theory of $\models$ to that of $\vdash$ is thus quite analogous to the passage from the study of the integers under addition to the consideration of the class of additive groups. But in this case, the theory of $\vdash$ at which we arrive is precisely what we call a formal deductive system. Those formulas θ for which we have postulated $\vdash\theta$ are called the *formal axioms* of the deductive system; those postulates about $\vdash$ having the form "whenever $\Gamma_1\vdash\theta_1$ and $\Gamma_2\vdash\theta_2$ then $\Gamma_3\vdash\theta_3$" define what we call the *formal rules of inference* of the deductive system. By combining formal axioms with formal rules of inference we obtain derived statements of the form $\vdash\theta$, e.g., we may derive $\vdash\phi\rightarrow\phi$ for all sentences ϕ of $\mathfrak{L}$; such sentences θ are called *formal theorems*, and the sequence of steps leading from axioms and rules to such a formal theorem is called a *formal proof*. Similarly, we may derive statements of the form "If $\Gamma_1 \vdash \theta_1$ then $\Gamma_2 \vdash \theta_2$," e.g., with formal axioms and rules of the usual kind we obtain the result that whenever $\Gamma \vdash \phi$ then also $\Gamma \vdash \psi \to \phi$. Such statements are called *derived rules of inference*. The derivation of such rules and of formal theorems constitutes the elementary part of deductive logic; it is entirely analogous to the derivation of identities holding in every group, from the group axioms.

(iv) Finally, we consider the totality of relations $\vdash$ (called *formal implications*) satisfying the postulates describing the formal axioms and rules of inference, and we seek to relate them to the relation $\models$ of logical inference from which we started. In the usual systems of elementary logic, for example, we are able to show that $\models$ is the intersection of all formal implications $\vdash$; we call this the *completeness* property of the deductive system, for it allows us to infer that whenever $\Gamma \models \phi$ then also $\Gamma \vdash \phi$ for every formal implication $\vdash$, and hence we can establish that $\Gamma \vdash \phi$ by using our formal axioms and formal rules of inference. The establishment of such a completeness result for a formal deductive system can thus be seen as a kind of representation theorem when the formulation of the deductive system is viewed as a process of algebraizing a theory of logical implication.

2. Another area of foundational work in which the algebraic aspects play an important role is in the theory of models. In this work we seek relations between the syntactical form of a given set of sentences, and the totality of structures which make all of these sentences true, i.e., which are models of the given set of sentences. Typically, these classes of models are characterized in terms of closure under algebraic operations for combining given structures to obtain new ones.

One of the earliest and most important examples of this type of work was the characterization of equational classes by Garrett Birkhoff. Suppose, for example, we are interested in structures of the form $(A, \circ, f)$, where A is a set closed under a 2-place operation $\circ$ and a 1-place operation f. (The totality of such structures is an example of a *similarity class*. This one we shall denote by S.) We consider a rudimentary language, $\mathfrak{L}$, equipped for discourse about any structure $(A, \circ, f) \in S$. $\mathfrak{L}$ is to contain variables $x, y, z, \ldots$ ranging over A, symbols $+$ denoting the operation $\circ$ and $-$ denoting the operation f, parentheses, and an equality sign. Using these symbols, we can form all sorts of equations, such as $x+y=y+x$, $-(x+y)=-y$, $--x=x+-y$, etc. Given any such equation, and any structure $(A, \circ, f) \in S$, we can inquire whether the equation holds identically in the structure or whether it fails when some elements in A are assigned as values of the variables. Finally, given any set Γ of equations we can consider the totality S_Γ of all those structures of S in which every equation of Γ holds identically. A class of structures determined in this way by a set of equations is called an *equational class*. The problem for the theory of models is to characterize the equational classes from among all subclasses of S. The solution found by Birkhoff is simple and elegant: A subclass of S is equational if, and only if, it is closed under formation of subalgebras, homomorphic images, and direct products.

These three methods of obtaining new structures from given ones first became familiar to mathematicians from the study of groups and rings, were then found of use in classifying other varieties of structures, and were finally recognized as having a natural setting in the general theory of algebraic structures (now considered a part of the foundations of mathematics). Although various older methods of combining algebraic structures have proved of use in the theory of models, this theory has also led to the consideration of new methods devised especially for its own ends. One of the most useful of these is the notion of an *ultraproduct* of a given family of structures.

Suppose we again study the similarity class S of structures $(A, \circ, f)$ considered above, and fix attention on some particular family $\{(A_i, \circ_i, f_i)\}_{i\in I}$ of them. The *direct product* of this family, as is well known, is the following structure $(B, \circ, f)$:

(a) B is the set of all functions ϕ, with domain I, such that $\phi_i \in A_i$ for all $i \in I$;
(b) for any $\phi, \psi \in B$, the element $\phi \circ \psi$ of B is such that $(\phi \circ \psi)_i = \phi_i \circ_i \psi_i$ for all $i \in I$;
(c) for any $\phi \in B$, the element $f\phi$ of B is such that $(f\phi)_i = f_i\phi_i$ for all $i \in I$.

An ultraproduct of the family $\{(A_i, \circ_i, f_i)\}_{i\in I}$ is a homomorphic image of the direct product $(B, \circ, f)$ obtained in a special way, as follows.

We consider filters on I. These are sets F of subsets of I, such that F is closed under $\cap$, and that $X \in F$ and $X \subseteq Y \subseteq I$ implies $Y \in F$. If a filter I is proper, i.e., $\varnothing \notin I$, then it can always be extended to a filter I' such that for *every* $X \subseteq I$ either X or its complement is in F, but not both. A filter I' of this kind is called an *ultrafilter*. Choosing any ultrafilter F on I we can pass from the direct product $(B, \circ, f)$ of the family $\{(A_i, \circ_i, f_i)\}_{i\in I}$, to a new structure $(B/F, \circ/F, f/F)$ as follows. We consider the relation $\equiv_F$ on B such that for any $\phi, \psi \in B$ we have $\phi \equiv_F \psi$, iff $\{i : \phi_i = \psi_i\} \in F$. This is easily seen to be an equivalence relation, and we take B/F to be the set of equivalence classes ϕ/F for all $\phi \in B$. Next, we find that whenever $\phi \equiv_F \psi$ we also have $(f\phi) \equiv_F (f\psi)$; hence f induces an operation f/F on B/F such that $(f/F)(\phi/F) = (f\phi)/F$ for each $\phi/F \in B/F$. Similarly, whenever $\phi_1 \equiv_F \psi_1$ and $\phi_2 \equiv_F \psi_2$, then we also have $(\phi_1 \circ \phi_2) \equiv_F (\psi_1 \circ \psi_2)$; hence we obtain an operation $\circ/F$ on B/F such that

$$(\phi/F)(\circ/F)(\psi/F) = (\phi \circ \psi)/F$$

for any $\phi/F, \psi/F \in B/F$. A structure $(B/F, \circ/F, f/F)$ obtained in this way from the direct product $(B, \circ, f)$ and any ultrafilter F on I is called an *ultraproduct* of the family $\{(A_i, \circ_i, f_i)\}$.

Ultraproducts are of interest in the theory of models when we pass from the equational language $\mathfrak{L}$ for our similarity class S to the first-order language $\mathfrak{L}'$ for S. In $\mathfrak{L}'$ we form sentences by first combining equations of $\mathfrak{L}$ by (repeated) use of the connectives $\neg$ (not), $\wedge$ (and), $\vee$ (or), $\rightarrow$ (if . . . then), $\leftrightarrow$ (iff), and then prefixing to the resulting formula Q a string of quantifiers, $\forall v$ (for all v) or $\exists v$ (for some v), one for each variable v occurring in Q. A class S_σ consisting of all structures $(A, \circ, f)$ of S for which some given sentence σ of $\mathfrak{L}'$ is true, is called an *elementary class*; an *elementary class in the wider sense* is a subclass S_Γ of S consisting of all structures satisfying some set Γ of sentences of $\mathfrak{L}'$.

Now the basic connections between ultraproducts and sentences of $\mathfrak{L}'$ are as follows:

(i) For any sentence σ of $\mathfrak{L}'$, any family $\{(A_i, \circ_i, f_i)\}_{i\in I}$ of structures of S, and any ultrafilter F of I, σ will be true of the ultraproduct $(B/F, \circ/F, f/F)$ of the family iff

$$\{i \in I : \sigma \text{ is true of } (A_i, \circ_i, f_i)\} \in F.$$

(ii) A subclass $T \subseteq S$ is elementary iff both T and $S \sim T$ are closed under formation of ultraproducts. T is elementary in the wider sense iff K is closed under ultraproducts and $S \sim T$ contains any ultraproduct of a family $\{(A_i, \circ_i, f_i)\}_{i\in I}$ such that $(A_i, \circ_i, f_i) = (A, \circ, f)$ for every $i \in I$, where $(A, \circ, f)$ is some element of $S \sim T$. (Such an ultraproduct is said to be an *ultrapower* of the structure $(A, \circ, f)$.)

(iii) Two structures $(A, \circ, f)$ and $(A', \circ', f')$ are called *elementarily equivalent* if there is no sentence σ of $\mathfrak{L}'$ which is true of one and false of the other. A necessary and sufficient condition for two structures to be elementarily equivalent is that there exists an ultrapower of one which is isomorphic to an ultrapower of the other.

The result (i) can be proved very simply from the definition of ultraproducts. Results (ii) and (iii), due to Keisler, have been proved under the Generalized Continuum Hypothesis. It is an outstanding open problem of the theory of models whether this hypothesis can be eliminated from the proof.

3. The last area of foundational work we shall mention in which algebraic aspects are predominant, consists of applications of results or methods from foundational studies to obtain particular results in some part of algebra. One part of algebra in which such results have been obtained is the theory of real closed fields. In 1948 Tarski published a decision method to determine in a finite number of steps whether any given sentence in the first-order language for ordered fields is true of the field of real numbers. This method depends on a basic lemma to the effect that a first-order sentence is true of the field of reals iff it is true in every real closed field. Tarski himself indicated many applications of his result to particular problems in algebra; but it was Abraham Robinson who showed how Artin's solution of Hilbert's 17th problem could be obtained very simply from Tarski's lemma: Every polynomial p which is definite (takes only nonnegative values) can be expressed as a sum of squares of rational

functions whose coefficients lie in the field generated by the coefficients of p. By appealing to the completeness theorem of first-order logic this result was strengthened by showing that the number of squares needed to represent a given definite polynomial depended only on its degree and the number of its variables.

The most widely known application of foundational methods to the solution of an algebraic problem is the work of Ax and Kochen on the fields Q_p (p-adic completion of the rationals). Artin conjectured that every form of degree d over Q_p, in which the number of variables exceeds d^2, has a nontrivial zero in Q_p. This conjecture had been established for $d=2$ and 3. Ax and Kochen showed that the conjecture is true for arbitrary d with the possible exception of a finite set of primes p (depending on d). Subsequently, it was found that the original conjecture is not true in full generality. The work of Ax and Kochen involves the use of ultraproducts and the study of elementary classes in the similarity class of rings. Subsequently Ax turned some of these techniques to the study of finite fields and was able to settle an outstanding foundational problem: He showed the existence of a decision method to determine automatically, in a finite number of steps, whether any given first-order sentence holds for all finite fields.

3. Set-theoretic aspects of foundations. The explicit study of sets in relation to the foundations of mathematics was begun independently by two men, Cantor and Frege, at about the same time and place—Germany in the 1870's. Cantor's principal achievement was to develop set-theory as a foundation for the study of infinite sets, generalizing the notions of cardinal and ordinal numbers to apply to them. Frege, proceeding downward instead of upward, aimed to explain the theory of natural numbers in terms of set-theoretic notions.

The work of both these men foundered on the paradoxes. Burali-Forti showed in 1897 that there is a pair of incomparable ordinals, just about the time that Cantor published a proof of the contrary result. Frege's fundamental work was in press in 1902 when he received a letter from Russell indicating how contradiction could be developed in his system by considering the set of all sets which are not elements of themselves, and inquiring whether this set is an element of itself.

Both the Burali-Forti and the Russell paradoxes may be traced to an unbridled use of the "comprehension principle," which states that to any condition expressed by a sentential formula $\sigma(x)$ with one free variable, we may associate a set G consisting of all those objects x which satisfy the formula. By taking $\sigma(x)$ to be "x is an ordinal," we are led to Burali-Forti's paradox; by taking $\sigma(x)$ to be "x is a set which is not an element of itself" we are led to Russell's.

How are we to regard these paradoxes? They have something in common with the schoolboy riddle: "What happens when an irresistible force meets an immovable body?" Of course the answer is that there cannot be both an irresistible force and an immovable body: We can tell that much just from the meaning of the words "irresistible" and "immovable." The question whether either such a force or such a body exists is one requiring empirical investigation or physical theory to answer. . . . In set theory we have seemingly irresistible forces, such as the operation of passing from a given set G to the set $G \cup \{G\}$, which seems to indicate that every set can be enlarged. On the other hand, the set of all sets appears like an immovable body, incapable of enlargement because of its comprehensive character. Seeing clearly that we cannot have both, which one do we reject?

The way in which such fundamental decisions were made was through the formulation of axiomatic theories of sets. The version most widely known was presented by Zermelo in 1908. In this version the irresistible force is firmly ensconced, since each set G can be enlarged to $G \cup \{H\}$, where H is a subset of G for which we can prove $H \notin G$. The immovable object is not present, the argument of Russell's paradox being used to show that no set can have all sets among its elements. Still, the theory is intended to be comprehensive, as it claims to be a theory of all sets whatever. Zermelo's system was clarified by Skolem and strengthened by Fraenkel. The resulting system, ZF, is widely employed.

In the same year, 1908, Russell published his theory of types. Couched in the language of logic (propositional functions, rather than sets, are the basic objects of study), and encumbered by a hierarchical theory of ramified types to which a mysterious axiom of reducibility was added, it was hardly recognized as a theory of sets. As subsequently clarified and simplified by Ramsey and Chwistek, however, the theory of types emerges as a theory of sets—much more modest in scope, however, than Zermelo's. In its simplest version it deals with a sequence of domains $D_0, D_1, \ldots$, where D_0 is the domain of "individuals" (nonsets) and D_{n+1} is the domain of all subsets of D_n for each natural number n. Of course there is no pretense that these domains contain all sets whatever, but there are enough sets to develop the theories of the classical number systems and geometric spaces, and indeed, just about all parts of mathematics except a full theory of transfinite cardinal and ordinal numbers. The theory of types, like ZF, seems to favor the irresistible force over the immovable body: given any set in one of the domains D_n, we can always find larger ones in D_{n+1}; there is no all-inclusive set.

An axiomatic theory in which an immovable body is present to resist any enlarging force was originally conceived by von Neumann, recast by Bernays, and put in final form by Gödel; we call it the vNBG system. In it the basic objects are called classes, and two sorts are distinguished: those which are elements of other classes and those which are not. (The former are called sets, the latter proper classes.) Thus there is a class V of all sets, but it cannot be enlarged to $V \cup \{V\}$ since V is not an element of any class and so $\{V\}$ cannot be formed.

Although both the systems ZF and vNBG are set forth as though they deal with all sets whatever, there is a way of looking at these axiom systems which makes each of them appear, like the theory of types, as furnishing a theory for only a fragment of the naive Cantorian universe of sets. This perspective is based on the notion of rank.

We assume that the $\in$ relation for sets is well founded—intuitively, that there are no infinite descending chains of sets $\ldots x_3 \in x_2 \in x_1 \in x_0$. An elegant formulation of this property, due to von Neumann, is that every nonempty set x has an element y having no element in common with x. Under this assumption we can associate with every set x an ordinal $r(x)$, the rank of x, by the recursive rule that $r(x)$ is the least ordinal greater than every $r(y)$ for all $y \in x$. Clearly $r(\emptyset)=0$; if we adopt the von Neumann definition of ordinals under which $0=\emptyset$ and, more generally, each ordinal turns out to coincide with the set of ordinals less than it, we find that $r(x)=x$ for any ordinal x.

Now it turns out that all of the axioms of the system ZF will be satisfied if we interpret the undefined term "set" in these axioms as ranging over all and only those sets x such that $r(x)$ is less than a certain ordinal θ_1. This ordinal θ_1 is the first nondenumerable cardinal which is "strongly inaccessible," i.e., (1) if x is any set such that it and each of its elements has cardinality less than θ_1, then the union of its elements has cardinality less than θ_1, (2) if x is any set with cardinality less than θ_1 then the set of all subsets of x also has cardinality less than θ_1, and (3) no cardinal smaller than θ_1, other than ω, possesses properties (1) and (2). In the case of the axiom system vNBG, a model satisfying its axioms can be obtained by using only sets x of rank $r(x) \leqq \theta_1$; the proper classes of the system will be of rank θ_1, the sets will be of lesser rank.

Of course within ZF (if it is consistent) one cannot prove that there *is* a nondenumerable inaccessible cardinal. Most mathematicians working with axiomatic set theory, however, are neo-Cantorians who believe that whatever collection of ordinals they succeed in describing, there is always a next larger ordinal. If we assume (what is also unprovable, assuming consistency), that there is *no* nondenumerable inaccessible cardinal among the ordinals of the ZF universe of sets, then "the next greater ordinal" will be strongly inaccessible.

From the perspective afforded by the notion of rank, the theory of types and the systems ZF and vNBG appear not as competitors for "the right way" to formulate set theory, but as descriptions of different-sized slices[2] of a Cantorian universe which is somehow too large to be described in its entirety. Of course once this is recognized, there is a natural impulse to go on describing increasingly large slices of the indescribable totality, and a good deal of recent work has been along these lines. One of the most interesting efforts involves the notion of measurable cardinals, which we shall describe briefly below.

Although voyages to distant ordinals may seem to many mathematicians as impractical as extra-galactic expeditions, we know that in principle they provide "practical" advantages. All of the set theories discussed above, and any

[2] Although ZF describes a "slice" of the Cantorian universe consisting of all sets of rank $<\theta_1$, the smallest inaccessible ordinal, Montague and Vaught have found that there is some ordinal $\alpha<\theta_1$ such that ZF also describes the "slice" consisting of all sets of rank $<\alpha$.

other in which the axioms can be enumerated in an automatic manner, are known from Gödel's incompleteness result to possess sentences which are true but unprovable from the axioms. Furthermore, these sentences are of a very simple kind—indeed by Matijasevitch's recent solution of Hilbert's tenth problem we now know that they can be put in the form of sentences asserting the non-existence of a solution to some given diophantine equation. However, if we pass from a given theory of sets to one with sets of higher rank, we can generally prove some of these diophantine statements which were unprovable in the original system.

While diophantine statements have a simple form, they tend to lose interest when they contain 50 or 60 variables. By contrast, there are some quite short sentences which are known to be undecidable in any of the systems discussed above (type theory, ZF, vNBG)—namely, the axiom of choice and the continuum hypothesis, proved consistent by Gödel in 1936 and independent by Cohen in 1963. Several mathematicians have shown that the addition of various axioms asserting the existence of larger ordinals will not result in a system in which these axioms become decidable. (Incidentally, Cohen's independence results have been re-derived by Scott and Solovay using a notion of Boolean-valued models of set theory in which algebraic and set-theoretic methods are beautifully combined for foundational studies.)

So far we have been discussing axiomatic theories of sets, but this is by no means the only part of foundational studies we intend to classify under "set-theoretical aspects." Let us mention two others.

1. Set theory and logic are closely interrelated. We have described above how deductive logic may be considered as an algebraization of the notion of logical consequence. But how is this relation originally defined? Since $\Gamma \models \sigma$ holds iff the sentence σ is true in every structure which satisfies each sentence of Γ, we are led to inquire under what conditions a given sentence is true for a given structure. This inquiry was shown by Tarski, in 1934, to have no precise answer for sentences of natural language, which is inherently paradoxical; but he gave a mathematical definition of truth for sentences of formalized mathematical languages by using basic notions of set theory. This definition, and the more general semantical notion of satisfaction of a sentential formula by a sequence of elements of the given structure, have played an important role in many subsequent investigations into undecidable theories, relative consistency proofs, and the comparative strengths of two given theories.

The basis of any language is a grammar, G, possessing various kinds of symbols from which sentences and other syntactical categories can be built up. The syntactical operations on formulas of G allow us to consider the grammar as a kind of algebraic structure. These structures are "free" in their similarity class. When we interpret the grammar G with respect to a mathematical structure S, we are essentially mapping the structure G homomorphically—not into S itself, but into a cylindric field of relations over S, generated by the fundamental operations and relations of S. Thus the set-theoretical notions involved in cylindric fields of relations, as well as those required to justify recursive definitions in free structures, are bound up with the definition of truth, and hence with the fundamental logical notion of consequence. Although, for first-order languages, this semantically-defined relation can be shown equivalent to a syntactically defined notion of derivability, it is known from Gödel's work on incompleteness that this duality cannot be extended to higher-order languages. Thus the semantical notions, rooted in set theory, provide our only access to the logic of higher-order languages.

2. In 1959, in Warsaw, an international symposium was held on infinitistic methods in logic. A few years before, logicians had begun to study the logic of formal languages differing radically from natural languages in that they contained infinitely long sentences. Sentences of such languages could be infinitistic in three ways:

(i) atomic sentences could consist of a relation symbol of transfinite rank followed by a transfinite sequence of individual symbols,

(ii) infinite sets of sentences could be combined into one by conjunction, $\sigma_1 \wedge \sigma_2 \wedge \sigma_3 \wedge \ldots$, or by disjunction, $\sigma_1 \vee \sigma_2 \vee \sigma_3 \vee \ldots$, and

(iii) quantification could be carried out over infinitely many variables arranged in arbitrary order type, e.g.,

$$\cdots \forall v_6 \exists v_4 \forall v_2 \exists v_1 \forall v_3 \exists v_5 \cdots \phi(v_1, v_2, \cdots).$$

At first these infinitistic languages were regarded as curiosities, but in 1960 they were used by Tarski and Hanf to solve an old and difficult problem of set theory —thereby opening a new wave of research based on axioms of infinity which carry us much farther into the transfinite than any earlier ones.

The problem involves the existence of measures on sets of various cardinality. We shall be concerned with measures m, defined on all subsets of a set A, taking only values 0 and 1, such that $m(A)=1$ and $m(X)=0$ for each one-element set $X \subseteq A$. If κ is a cardinal, m is called κ-additive if $m(\bigcup_{i\in I} X_i) = \sum_{i\in I} m(X_i)$ whenever the family $\{X_i\}_{i\in I}$ of subsets of A has cardinality $<\kappa$. Of course if A is denumerable, i.e., of power ω, it has a measure m which is κ-additive for every $\kappa \leqq \omega$. For every nondenumerable cardinal α less than θ_1, the first nondenumerable inaccessible cardinal, it was known that a set A of cardinality α cannot have a κ-additive measure except for the case $\kappa \leqq \omega$. (In particular, it has no countably additive measure.) It was suspected that θ_1 would resemble the smallest inaccessible cardinal, admitting κ-additive measures for every $\kappa \leqq \theta_1$. Tarski, however, using Hanf's result about certain infinitary languages, was able to show that sets of cardinality θ_1 admit κ-additive measures only for $\kappa \leqq \omega$, hence have no countably additive measures.

Hanf's result deals with the compactness question of logic. For first-order languages of the ordinary kind, with sentences of finite length, the compactness theorem states that if Γ is a set of sentences such that the sentences of every finite subset of Γ are true in some structure, then there is a model in which *every* sentence of Γ is true. Hanf considered infinitary languages L_α for each cardinal α, in which one can form the conjunction of any set of sentences whose cardinality is $<\alpha$, and one can quantify simultaneously over any set of variables whose cardinality is $<\alpha$. For each α satisfying $\omega<\alpha\leqq\theta_1$, he produced a set Γ of sentences of L_α such that Γ has cardinality α, the sentences of each subset of Γ of cardinality $<\alpha$ are all true in some structure, but there is no structure in which every sentence of Γ is true.

Hanf extended this result far into the transfinite. For example, if $\theta_0, \theta_1, \ldots, \theta_\xi, \ldots$ is the sequence of all inaccessible cardinals, then θ_1 in the above theorem can be replaced by any θ_ξ such that $\xi<\theta_\xi$. For any α less than such a θ_ξ Tarski's result also holds: A set of cardinality α admits no countably additive, 2-valued measure defined on all of its subsets.

These methods of proof seem to offer no hope of extending the result to *every* cardinal $\alpha>\omega$. Accordingly, various mathematicians have begun to explore the consequence of assuming, as a very strong axiom of infinity, that there is a cardinal $\alpha>\omega$ which is measurable, i.e., admits an α-additive measure. For example, Scott has found that Gödel's axiom of constructibility is incompatible with the existence of a measurable cardinal.

4. Constructive aspects of foundations. Intuitionism is generally regarded as the most radical constructivist position in the foundations of mathematics. Its radicalism consists in the rejection of large parts of mathematics which are accepted by most mathematicians. For instance, the law of the excluded middle, allowing us to assert for any statement p that either p or not-p must hold, is rejected in contexts where infinitely many objects are under discussion, except for particular statements p, where we have a method to decide whether p holds, or whether not-p holds. Again, no argument is accepted for a statement of the form "there is an integer p such that $\sigma(p)$," short of a proof which furnishes a specific way of arriving at a particular integer p such that $\sigma(p)$ can be established.

While it is not hard to ascribe these rejections to a desire for a more constructive approach to mathematics, it seems that the radical changes sought by intuitionists have other motivations, too. One of these is bound up with the name "intuitionism," and relates to the fundamental question of the significance of mathematical discourse. For the "classical" mathematician such discourse serves to communicate "facts" about certain abstract objects such as numbers, sets, and points. For the intuitionists, however, the aim of mathematics is "mental mathematical construction," and discourse is directed toward communication about this kind of intuitive experience. Such a disparity as to the purpose of mathematical discourse is sharply indicated by the following contention of some mathematicians about the intuitionist position: that a certain mathematical statement, e.g., about the existence of a number with a certain arithmetical property, may be true at certain times but not at others, or true for some mathematicians but not for others. I myself am unsure whether intuitionists agree with this contention—perhaps some do and others don't. . . . At any rate it is certain that the intuitionistic and the classical mathematician use language in very different ways, and it seems evident from this fact that clear communication between them is not to be expected, unless and until they can agree on what constitutes a satisfactory translation from the language of one to that of the other.

One of the complaints by classical mathematicians about intuitionists is that the latter have been unwilling to commit themselves to clear-cut criteria for judging the correctness of a mathematical argument by furnishing an explicit list of axioms and rules of inference—i.e., a formal deductive system. As a gesture toward improving communication, Heyting—the leading exponent of intuitionism after Brouwer—published a paper in 1930 setting forth formal rules of intuitionistic propositional logic. All these rules are included among the classical laws of logic, but many of the latter are not derivable in Heyting's system. For instance, $p \rightarrow \neg\neg p$ is in, $\neg\neg p \rightarrow p$ is out. Heyting emphasizes, however, that while the rules of his system can be established intuitionistically, no formal system can be proved to exhaust the totality of laws which may be established by further intuitionistic constructions. It should also be noted that whereas classically the laws of logic come first, in that they are used in the development of other parts of mathematics, the intuitionist starts with arithmetic—constructions with the whole numbers—and only arrives at laws of logic as a kind of generalization of universally-valid methods of arithmetical construction.

Heyting's system of intuitionistic propositional logic has been the subject of various foundational studies pursued by means of classical mathematical methods. First Gödel showed that even though the Heyting system was on the face of it weaker than classical propositional logic, one could nevertheless associate with each propositional formula σ another one, σ', such that σ is classically valid if and only if σ' is provable in the Heyting system.

Tarski showed how a set-theoretical interpretation could be provided for the Heyting system. Classically, we associate with each formula σ of propositional logic a function σ_c of several arguments ranging over the subsets of an arbitrary set U. (The function σ_c is defined recursively; for example, for any given arguments the value of $(\neg\sigma)_c$ is the complement of the value of σ_c with respect to U.) Then a formula σ will be classically valid iff σ_c is the constant function with value U. In Tarski's interpretation, with each formula σ one associates a function σ_i whose arguments and values range over the open sets of an arbitrary topological space U. (For example, the value of $(\neg\sigma)_i$ is the interior of the complement of the value of σ_i, for any given arguments.) The formulas provable in Heyting's system are just those with constant value U in the Euclidean plane.

The most important methodological analysis of intuitionistic mathematics by classical methods was undertaken by Kleene. His tool was the recursive function, originally developed as a mathematically precise notion equivalent to that of automatically calculable function, for the purpose of dealing with decision problems. Using this class of functions, Kleene defined a notion of realizability for sentences of arithmetic, which he proposed as an intuitionistic counterpart to the classical notion of truth. Kleene was able to show that the sentences realizable by some number were just those provable in Heyting's system of intuitionistic arithmetic. Attempts to extend this approach to intuitionistic logic were not as successful.

Recursive functions have been used in various ways to provide constructive alternatives to portions of classical mathematics—alternatives quite different from those afforded by intuitionism. For example, there is *recursive analysis*. The basic objects are real numbers treated as infinite decimal expansions, but one admits only those numbers whose decimal parts admit some rule by which one can recursively, i.e., automatically, compute the digit in the nth place, for any given n. Again, in dealing with real-valued functions of a real variable, one admits only those functions which can be computed in a recursive way for given arguments. This type of analysis, vigorously pursued a few years ago, has lost steam since the discovery that there was more than one way to limit the usable functions by recursive requirements that seemed intuitively plausible, but these ways were inequivalent.

There is also recursive set theory. In ordinary set theory two sets have the same cardinality when there is a one-one mapping of one onto the other. In the recursive version one deals only with sets of integers, and limits oneself to one-one mappings which are (partial) recursive. The resulting "cardinal numbers" are called isols, and their arithmetic has been extensively studied. . . . Another part of recursive set theory deals with recursive degrees of unsolvability. Where A and B are sets of natural numbers, we say that A is recursive in B if the question whether any given number is in A can be reduced by a recursive method to a finite number of questions about certain numbers being in B. A is recursively equivalent to B if each is recursive in the other; the corresponding equivalence classes are the degrees of unsolvability; the partial order induced on them by the relation A is recursive in B then determines a semi-lattice whose structure has been extensively studied.

It is interesting to note that though the study and application of recursive functions is rooted in finitistic approaches to mathematics, recent work has seen an impulse to extend this notion from functions on natural numbers to those which have a transfinite ordinal domain. Carried far enough, this theory makes contact with the universe of constructible sets created by Gödel for proving the consistency of the continuum hypothesis—a domain seemingly far removed from the finite.

5. Concluding remarks. Although we began by saying we would attempt to write a paper about the foundations of mathematics in the large, it should be very clear that we have not attempted to be all inclusive. Many important areas have gone unmentioned—we may cite the theories of categories, of hierarchies, of decision problems, and automata, to name but a few. We have, however, tried to touch upon sufficiently many broad areas to give the reader some sense of the extent of the subject, and more particularly to illustrate our thesis that these materials can be roughly classified according to algebraic, set-theoretical, and constructive aspects of the subject. These aspects are not competitive, but supplement each other to illuminate fundamental problems from different perspectives. There are many interrelationships among the subjects we have mentioned; only a few have been made explicit, partly because of the author's ignorance and in part because these relations are still only obscurely understood. We hope that readers with varying mathematical interests may be led to clarify some of these questions.

Covering as much ground as we have done, it would be easy to append an enormous bibliography of related works. The reader who would like to see that kind of a list is invited to inspect volume 26 of the Journal of Symbolic Logic, given over entirely to a bibliography of work in that field during the preceding 25 years. Instead, we have chosen a very small list of works where the non-specialist reader may obtain more detail about some of the topics we have mentioned. The bibliographies of these works will then lead further into the subject for those who still have interest.

A limited number of copies of all the Chauvenet Symposium Papers bound as a single volume may be obtained by writing to Professor J. C. Abbott, Department of Mathematics, U. S. Naval Academy, Annapolis, Md., 21402.

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PATTERNS OF VISIBLE AND NONVISIBLE LATTICE POINTS

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1. Introduction. Let L_k, for $k \geqq 2$, denote the k-dimensional lattice, i.e., the set of points $(x_1, x_2, \cdots, x_k)$ with integral coordinates x_λ. A point in L_k will be called *visible* (namely, visible from the origin) if and only if its components x_λ have no common divisor greater than 1. Otherwise, the point will be called *nonvisible*. (By this definition, the origin itself is counted among the nonvisible points, a choice which we make purely for the sake of convenience.) In our geometrical representations we shall frequently use the following symbols for the points of L_k:

visible point = circle (○), nonvisible point = cross (×).

As a mnemonic aid we note that the first vowel both in visible and circle is an "*i*"; the first vowel both in nonvisible and cross is an "*o*".

For instance, Fig. 1 gives the distribution of circles and crosses among the points (x, y) of L_2 with $0 \leqq x \leqq 10$, $0 \leqq y \leqq 10$.

By a *pattern* P_k we mean the following: to each of the w^k lattice points $(x_1, x_2, \cdots, x_k)$ with $1 \leqq x_\lambda \leqq w$ is associated either a circle or a cross or neither

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