Enhanced Linking Numbers

Charles Livingston

1. INTRODUCTION. The study of knots and links begins with simple intuitive problems but quickly leads to sophisticated mathematics. This paper will provide the reader with an accessible route that begins with basic knot theory and leads into interesting realms of modern research. The specific topic of the paper is the *enhanced linking number*, λ, a new invariant of links that provides a simple tool to address some of the fundamental problems in the study of linking.

The linked rings illustrated on the left in Figure 1 are clearly linked; when a magician pulls such a link apart we know we have been tricked. A mathematical proof that this link is nontrivial depends on showing that its *linking number* is 1, where the linking number is the basic invariant of link theory, first described by Gauss two hundred years ago. With the link on the right the situation is less clear. If this link is built from rope or beads, then one quickly finds that unlinking the two is impossible. However, formulating a mathematical proof of this is more difficult; for instance, the linking number turns out to be of no help. The enhanced linking number reduces the proof that this link is nontrivial to a simple calculation. This article describes the enhanced linking number and shows how it can be applied to this basic problem of distinguishing links. It will also discuss the role of λ in a variety of advanced topics, such as periodicity, symmetry, Brunnian linking, and the modern theory of finite type invariants.

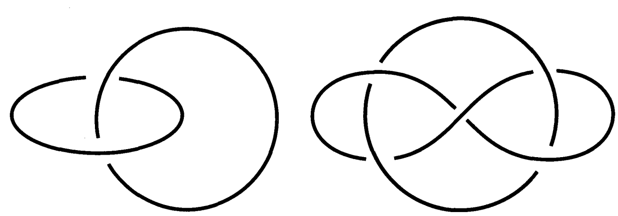


Figure 1. Two links.

Traditionally, knot theory has focused on the study of *knot invariants*. In most cases a knot invariant is simply a function that assigns to each knot or link an integer. Recently there has been a shift in knot theory from studying knot invariants themselves to examining how the invariants change as the knot or link is changed. The situation is somewhat parallel to a move from studying real functions to the study of derivatives. In the case of a knot invariant, the notion of derivative is replaced by a *crossing change formula*. This new focus on crossing change formulas in knot theory will be demonstrated by our presentation of the enhanced linking number. The presentation begins with a description of the crossing change formula for λ, and from this all the results of the article will follow. It will only be in section 8 in which the actual definition of λ is presented.

This article is largely self-contained. Other expository articles on knot theory that are related to this one include [**2**], [**6**], [**9**], [**11**], and [**13**] and books on the topic include [**7**] and [**10**]. The enhanced linking number was first described in a paper written by the author and Paul Kirk [**8**]. It generalized an invariant of linking number 0 links first described explicitly in [**12**] and now referred to as the Sato-Levine invariant.

2. KNOTS, LINKS, AND THE REIDEMEISTER MOVES.

Definition of links. A *link* is a collection of disjoint closed curves in R^3 without self-intersections. Figure 2 illustrates four links, each with two components. A link with one component is called a *knot*. The main focus in this article will be links of two components, which will be denoted as *ordered* pairs (K, J) of knots. This study will also be facilitated by considering *oriented* links, those for which a direction is assigned to each component, as indicated by the arrows in Figure 2. For later reference, three of the links in Figure 2 have standard names: U_2 is the *unlink*, H is the positive *Hopf link*, and W is the *Whitehead link*. (These last two are named for the topologists Heinz Hopf and J. H. C. Whitehead.)

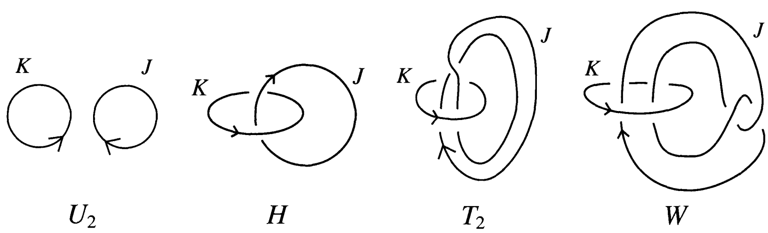


Figure 2. Basic two-component links.

Two different links with the same number of components are considered equivalent if one can be obtained from the other through a continuous deformation in which no component intersects another component or has a point of self-intersection. Figure 3 illustrates some of the steps of a deformation of a slightly complicated looking link into the unlink U_2.

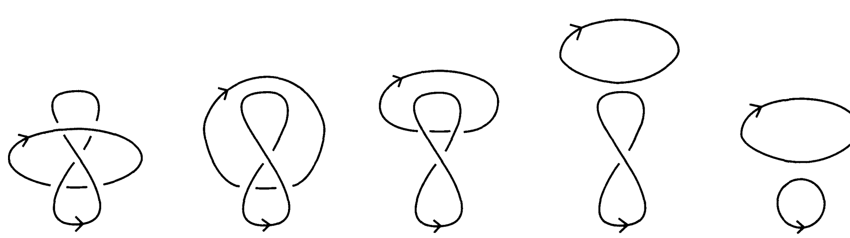


Figure 3. A deformation of a link to the unlink.

In section 3 the concept of linking number will be used to prove, for example, that U_2 and H are not equivalent.

Link diagrams. The illustrations of links in the previous figures are simply schematic pictures of links; these pictures reside on a flat, 2-dimensional, piece of paper and not in R^3. The relationship between links and their pictures is made formal with the notion of a *link diagram*.

A link itself sits in R^3 and hence can be projected onto a plane in R^3, for instance via the map that takes the triple (x, y, z) to the pair (x, y). The left diagram in Figure 4 represents a projection of a knot. In general, it is impossible to reconstruct the original link from its projection: in forming the projection too much information has been lost.

A *link diagram* is an illustration of the projection of a link with small gaps placed in the illustration to indicate "over and under" information about the link. The diagram on the right in Figure 4 illustrates the diagram of a knot having the projection shown on the left. Since links are oriented, diagrams also require orientations, which are indicated by small arrows, as has been done with the right-hand figure here.

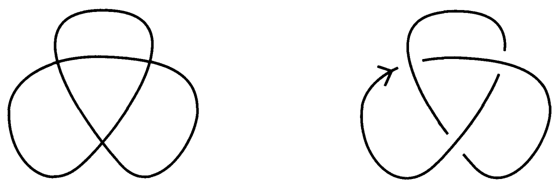


Figure 4. A projection and diagram.

Different links can have the same diagram. For instance, if the entire link is moved upward (in the z-direction), the diagram is unaltered. However, this upward shifting of a link is a deformation, and the new link is equivalent to the first. In general there is the following result, whose proof is a bit technical but not deep.

Theorem 2.1. *If two links have the same diagram, the links are equivalent.*

It is this theorem that makes formal the idea that links can be described using planar diagrams, as has been done beginning with Figure 1 in this paper.

Reidemeister moves. Since links are determined by link diagrams, a question immediately arises: How are the diagrams of equivalent links related? To understand the issue, look back at Figure 3. This figure is a sequence of link diagrams showing steps in the conversion of a diagram for one link into the diagram for the unlink. Notice that in each step of the conversion of the diagrams fairly simple changes take place. In fact, only three basic types of alterations of link diagrams are required. Theorem 2.2 states that, in fact, these simple moves, called the *Reidemeister moves*, are sufficient to convert any diagram of a link to any other diagram of any equivalent link.

Theorem 2.2. *Two diagrams represent equivalent links if and only if the diagrams are related by a sequence of Reidemeister moves, as illustrated in Figure 5.*

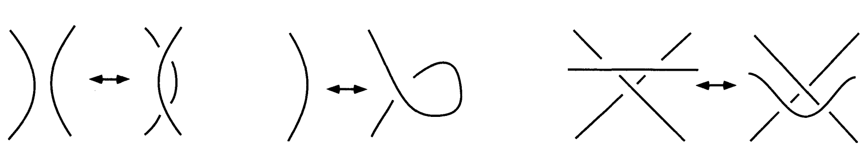


Figure 5. The six Reidemeister moves.

With this theorem it becomes possible to define a link invariant using a diagram for the link. To check that the invariant is well-defined, one needs to verify only that performing a Reidemeister move does not change the value of the invariant. This will be illustrated with the linking number in the next section. Further discussions of the Reidemeister moves can be found in such expository articles as [**2**], [**6**], or [**11**].

For practice with the Reidemeister moves, the reader might find the sequence of moves that correspond to the deformation in Figure 3. (Calling the moves illustrated in Figure 5 $R1$, $R2$, and $R3$ in order from left to right, the answer to this problem is the sequence $R1$, $R3$, $R1$, $R2$.)

3. LINKING NUMBERS OF TWO-COMPONENT LINKS. The simplest and most intuitive invariant in knot theory is the linking number. This invariant assigns to each link of two components an integer that roughly measures how many times one component "passes through" the other. Before giving a precise description of the linking number, we use the four examples of Figure 2 to illustrate the concept. The linking numbers of the first three links (U_2, H, and T_2) in Figure 2 are 0, 1, and 2, respectively. The fourth link is the most interesting. It has linking number 0; J appears to "pass through" K twice, but because it goes through in opposite directions, in computing the linking number the total contribution is 0. In particular, the unlink and the Whitehead link both have linking number 0. The first application of the enhanced linking number will be to show that the Whitehead link is not the same as the unlink. More formally, W and U_2 are not equivalent links.

An explicit method for computing the linking number of a link (K, J) is given by computing the sign of each crossing in a diagram for the link, as now described. A *crossing point* in a link diagram is a point on the diagram that corresponds to a double point in the link projection. The diagram in Figure 6 has five crossing points, indicated by large "dots" in the figure.

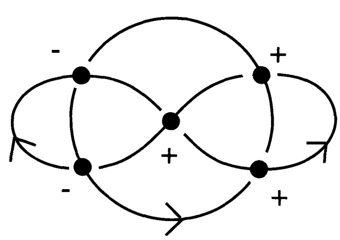


Figure 6. Crossing points and signs of crossings.

A *sign of the crossing* is associated to each crossing point as follows. The portion of the link that projects into a small neighborhood of the crossing point in the plane consists of two short segments, one that passes over the other, such as in the portions of link diagrams in Figure 7. View the link from the perspective of someone sitting on the upper of the two segments, facing in the direction of that component of the link. The sign of the crossing is *positive*, and the crossing is *right-handed*, if the lower segment crosses under the upper one moving from right to left. Similarly, it is called *negative* or *left-handed* if the lower segment passes under from left to right. Figure 7 presents

Figure 7. Right- and left-handed crossings.

"generic" pictures of right- and left-handed crossings. In Figure 6 the signs of each crossing are indicated with plus and minus signs.

Definition 3.1. The *linking number* $\mathrm{lk}(K, J)$ of a link (K, J) is given by the sum of the signs of the crossing points of a diagram for (K, J) at which K crosses over J. In this sum, right-handed crossings are counted $+1$ and left-handed crossings are counted -1. Self-crossings of K and J are not included in the summation.

The computation of the linking number of a link appears to depend on the link diagram. However, there is the following result and its immediate corollary.

Theorem 3.2. *If two link diagrams represent equivalent links, the linking numbers computed from the two diagrams are the same.*

Corollary 3.3. *Equivalent links have the same linking number.*

The proof of Theorem 3.2 is an easy exercise using the Reidemeister moves. For instance, the first Reidemeister move introduces (or removes) two crossing points in the diagram. However, the signs of these crossings will be opposite, so they cancel in the sum used to compute the linking number. In the second Reidemeister move only one crossing point is introduced, but this is either a crossing point between K and itself or J and itself, so it doesn't contribute to the sum. The complete proof is left to the following exercises, included to assure that the concept of linking number is well understood.

Exercise 3.4.

1. Check that performing any of the Reidemeister moves on a two-component link diagram does not change the count given in the forgoing algorithm for computing the linking number, and hence show that the linking number is well-defined, as claimed in Theorem 3.2. (Notice that for the third move there are many cases, depending upon the way in which the components of the link are involved in the Reidemeister move.)
2. The sign of the linking number changes if the orientation of one component is reversed.
3. The linking number is symmetric, $\mathrm{lk}(K, J) = \mathrm{lk}(J, K)$. Hint: Deform the link by rotating it 180 degrees about the vertical axis in the plane of the page. This will switch overcrossings of K with overcrossings of J.

The crossing change formula for the linking number. If a particular crossing point is selected in a link diagram, one can construct a new link diagram by making what is called a *crossing change*. On the left in Figure 8 a diagram of the Hopf link is drawn,

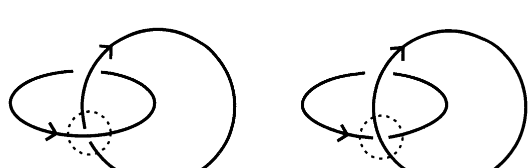


Figure 8. Hopf link before and after crossing change.

with a small dotted circle drawn around one of the crossing points. On the right in Figure 8 the same diagram is drawn, only modified so that the portion of the link that went over (within the dotted circle) is now going under. In general, such a modification, at any crossing point in a diagram, is called a *crossing change*.

Notice that performing this crossing change on the diagram resulted in a diagram of a new link that is *not* equivalent to the original one. Unlike Reidemeister moves, crossing changes do not correspond to deformations. In Figure 8 the crossing change converted the Hopf link into an unlink. There is a simple formula describing how the linking number of a link changes when a crossing change is performed on its diagram. If at the crossing point one component passes over itself, the linking number is unaltered. However, if at the crossing point one component passes over the other component, then the linking number does change. This change is encapsulated by the following equation, the *crossing change formula for the linking number*:

$$\mathrm{lk}(⤱) - \mathrm{lk}(⤲) = 1.$$

This equation uses very condensed notation. The symbol ⤱ is a stand-in for an entire link diagram, with the ⤱ representing only that portion of the diagram near a crossing point at which the crossing change is to be made. The slightly changed symbol ⤲ is a stand-in for a second link diagram, identical to the first except that the one crossing change has been made. Small dots are placed on segments in the illustrated portions of the link diagram. Placing one dot on one segment and two on the other is used to indicate that they come from different components of the link. Later there will be diagrams in which the two segments have the same number of dots, indicating that they come from the same component of the link. There will also be diagrams with no dots, in which case it is irrelevant which components the segments correspond to. As an illustration of the use of the crossing change formula, consider the Hopf link H on the left in Figure 8. In this case the diagram has been changed in that a right-handed crossing between the two components has been replaced by a left-handed crossing. The crossing change formula then tells us that the difference of the linking number of the Hopf link and the unlink is 1. In particular, these are different links. Of course, in this case computing the linking numbers is a triviality and the crossing change formula is not really needed.

The truth of the crossing change formula for the linking number is almost self-evident. If exactly one crossing change is made and it involves both components of the link, then the count that gives the linking number will change by ± 1, the sign depending on the sign of the crossing.

We have introduced the crossing change formula here not so much for its own usefulness, but to illustrate the general notion of crossing changes and crossing change formulas, and also to introduce the compact notation that is essential in this work.

The Fundamental Theorem. The following result will be the basis of the use of a crossing change formula in developing properties and computing the value of the enhanced linking number.

Theorem 3.5. *If links L_1 and L_2 have the same linking number, then there is a sequence of Reidemeister moves and crossing changes that converts the diagram of L_1 into the diagram of L_2, where none of the crossing changes in the sequence is between different components of the link.*

Proof. The argument is a bit technical to present in detail, but the ideas are straightforward and easily carried out in practice. The proof consists in showing that each L_i can be converted, using crossing changes and deformations, into a standard form. The first step of the proof is based on the observation that by changing crossings in one of the components it can be arranged that this component is unknotted, meaning that it can be deformed to a standard circle having no self-crossing points. After making these crossing changes and performing such a deformation, a link such as the one illustrated in Figure 9 results, in which one of the two components is obviously unknotted.

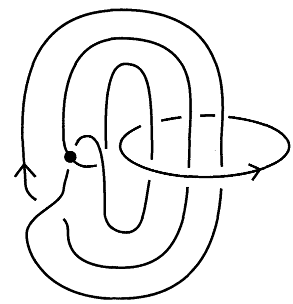


Figure 9. The link after unknotting one component.

It is now possible to make crossing changes in the second component so that it winds "monotonically" about the unknotted component. We can clarify this notion using the diagrams in Figures 9 and 10. In Figure 9 the curve, on the plane of the paper, has four maximum points with respect to the vertical, y-axis. The linking number for this link is 1. However, in the diagram in Figure 10 the curve representing the second component (the more complicated looking of the two components) has five maxima, and the link also has linking number five. In general, we say that the second component winds monotonically about the first, unknotted, component if the number of maximum points on the diagram equals the absolute value of the linking number. So in Figure 9 one component does not wind monotonically about the other, but changing the crossing marked with the dot and performing a sequence of Reidemeister moves changes it into a link diagram for which the second component does wind monotonically around the first. As a final step one shows that a further sequence of crossing changes and Reidemeister moves results in a link diagram in the "standard" form of T_n, $n \geq 0$, illustrated in Figure 10 for $n = 5$. The general T_n is analogous to this. ■

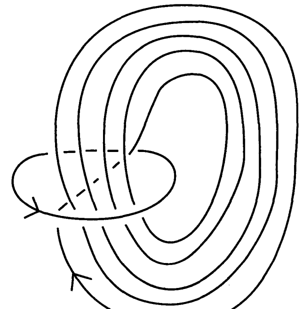


Figure 10. Standard form T_n, $n = 5$.

4. THE ENHANCED LINKING NUMBER. The linking number is a very useful, but also very simple invariant. The challenge and interest in studying linking becomes much greater when one takes on the problem of working with links of the same linking number. In this section the *enhanced linking number*, denoted λ, is introduced.

The formal definition of λ, along with an algorithm for its computation, will be delayed until section 8. For this reason we want to be very clear now: λ is an integer-valued function of two-component links. That is, for each two-component link (K, J) the invariant λ returns an integer, denoted $\lambda(K, J)$. Also, the value of λ will ultimately be seen to be the same for equivalent links. That is, if (K', J') is obtained by deforming (K, J), then $\lambda(K', J') = \lambda(K, J)$. The reader might at first be disappointed that much of this paper is based on the theory of an invariant whose definition and computation is a mystery. But this is one point of the article. *To apply λ to both computational and theoretical problems in link theory, one needs to know only a crossing change formula satisfied by λ—its precise value need not be known.* This crossing change formula is our next topic.

The crossing change formula for $\boldsymbol{\lambda}$. The following theorem states a number of facts. First it asserts the existence of the function λ (a fact that will be proved in section 8) with domain the set of all two-component links and range the set of integers. In particular, though we will initially be working with link diagrams, implicit in this is that the value of λ will be the same on equivalent links. The theorem also asserts that λ satisfies a particular crossing change formula. This formula is stated in a fairly condensed form that will be explained in detail following the statement of the formula. Hopefully, working with the notation of the crossing change formula for the linking number will have prepared the reader for this more complicated set-up.

Theorem 4.1. *There is an integer-valued invariant λ of two-component links satisfying the crossing change formula given in Definition 4.2.*

Definition 4.2. The crossing change formula for λ is given by:

$$\lambda\left(\diagup\!\!\!\!\diagdown, J\right) - \lambda\left(\diagdown\!\!\!\!\diagup, J\right) = \mathrm{lk}\left(\rangle, J\right)\mathrm{lk}\left(\langle, J\right). \qquad (1)$$

Figure 11 will help explain this notation. If one begins with a knot diagram K and focuses on a particular right-handed crossing, the resulting diagram has been abbreviated $\diagup\!\!\!\!\diagdown$. If the $\diagup\!\!\!\!\diagdown$ in that diagram is replaced with a $\diagdown\!\!\!\!\diagup$, a crossing change has been made. Another option is to replace the $\diagup\!\!\!\!\diagdown$ with $\rangle\langle$, in which case a link of two components is constructed. This link is illustrated in the third diagram of Figure 11.

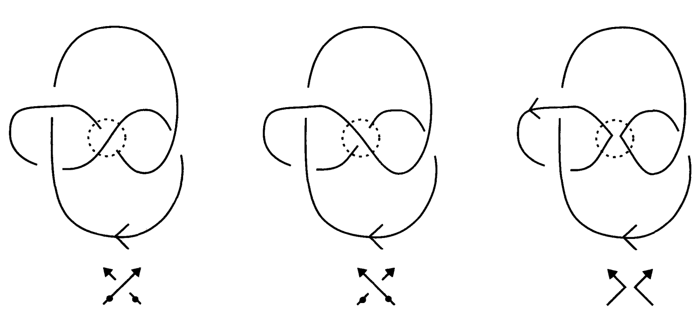
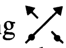
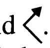
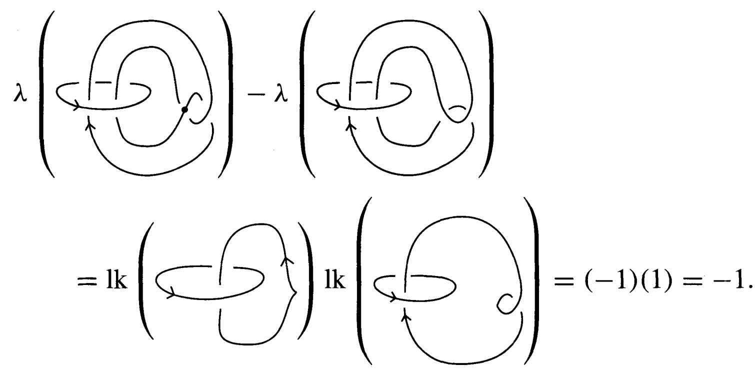


Figure 11. Smoothing a crossing.

The process of replacing  with  is sometimes called *smoothing the link* at the crossing. Since the smoothed link has two components, the two components are abbreviated in the formula by  and . The order will be seen not to matter.

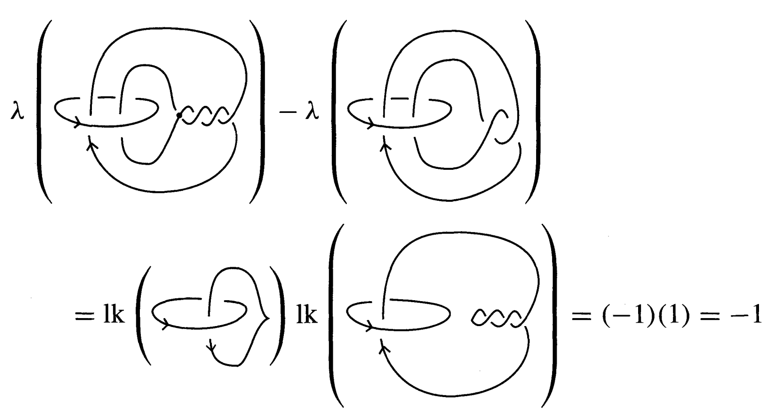
With the notation explained, it is best to illustrate the crossing change formula for λ with a series of examples, the first of which concerns the Whitehead link W in Figure 2. The application of the crossing change formula results in equation (2). Observe the essential feature in this equation that in each case λ is being applied to a two-component link. In this example the crossing change and smoothing are taking place near the crossing marked with the dot.



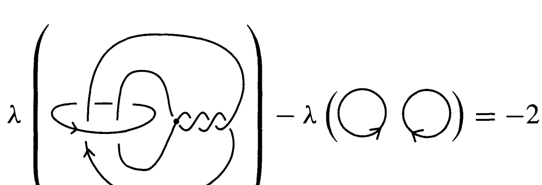
(2)

Notice that the second diagram in this difference is a diagram for the unlink U_2. Hence, since the value of λ on the unlink and the Whitehead link are different, the Whitehead link cannot be deformed into the unlink. The reader is encouraged to check the details of this calculation.

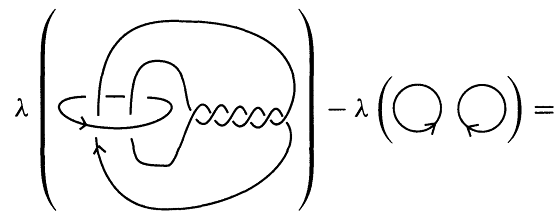
The following offers another example, working at the dotted crossing.



Adding this to the difference given in (2) yields



Repeating this process, one then finds that

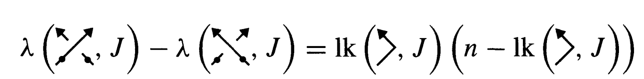


$$= -3.$$

Generalizing this example provides an infinite family of distinct links, all with linking number 0, distinguished by the value of λ. Notice that at this point we have been able to use λ to distinguish an infinite family of links, all with linking number 0, without ever computing the value of λ explicitly, or even knowing the definition of λ. All that has been used is the knowledge that the function λ exists and that it satisfies the crossing change formula. A few exercises might help the reader at this point.

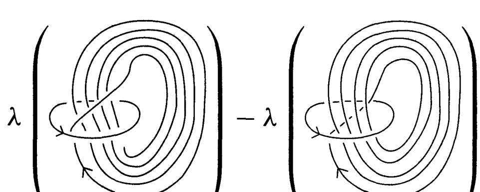
Exercise 4.3.

1. Show that the crossing change formula can be written as



   where $n = \mathrm{lk}(K, J)$.
2. Suppose that links (K, J) and (K', J') have the same linking number and that $\lambda(K, J) - \lambda(K', J') = n$. New links can be formed by reversing the orientations of K and K', call them (K_r, J) and (K'_r, J'). Show that $\lambda(K_r, J) - \lambda(K'_r, J') = n$.
3. From a given link (K, J) a new link can be formed as the mirror image: change all the right-handed crossings into left-handed crossings, and conversely. Denote the new link by $\overline{(K, J)}$. (Observe that this changes the sign of the linking number.) Use the crossing change formula to show that if a diagram for (K, J) can be converted into a diagram for (K', J') by making crossing changes at self-crossing points, then $\lambda\overline{(K, J)} - \lambda\overline{(K', J')} = -(\lambda(K, J) - \lambda(K', J'))$.

Basic examples: the links T_n. A somewhat more complicated application of the crossing change formula for λ is presented in the following computation. The two links that appear in it will be useful in later sections. The one on the right is T_5, as introduced in Figure 10; the one on the left will be denoted $\overline{T}_5$. (It turns out that T_5 is the more important of the two; in Theorem 8.8 it will be seen that $\lambda(T_5) = 0$.)



$$= 1 \cdot 4 + 2 \cdot 3 + 3 \cdot 2 + 4 \cdot 1 = 20.$$

To understand this, we momentarily denote $\overline{T}_5$ by L_0. To change $\overline{T}_5$ into T_5, four crossing changes are required; in the diagram these are at four self-crossings of the second component of $\overline{T}_5$. Each crossing change creates a new link; let's denote them L_1, L_2, L_3, and $L_4 = T_5$. We now have that

$$\lambda(L_0) - \lambda(L_4) = [\lambda(L_0) - \lambda(L_1)] + [\lambda(L_1) - \lambda(L_2)] \\ + [\lambda(L_2) - \lambda(L_3)] + [\lambda(L_3) - \lambda(L_4)].$$

The crossing change formula is now used to compute each of these summands. For instance, the first crossing change, say the lower left one, changes λ by $1 \cdot 4$. Performing the next changes λ by $2 \cdot 3$. Performing the last two produces changes of $3 \cdot 2$ and $4 \cdot 1$, respectively. The total of these changes is 20.

If one considers the analogs $\overline{T}_n$ and T_n of linking number n instead of the specific case of $n = 5$ just presented, a straightforward algebraic calculation yields the formula

$$\lambda(T_n) - \lambda(\overline{T}_n) = \sum_{j=1}^{n-1} j(n-j) = \frac{n^3 - n}{6}. \tag{3}$$

The reader is invited to carry out the calculation, recalling the formulas

$$\sum_{j=1}^{N} j = \frac{N(N+1)}{2}, \qquad \sum_{j=1}^{N} j^2 = \frac{N(N+1)(2N+1)}{6}.$$

5. SYMMETRY. Consider the unoriented links illustrated in Figure 12. The second of the two links is formed by taking the mirror image of the first, that is, by changing every crossing in the diagram. It turns out that the second link is a deformation of the first. To see this, consider the two step deformation illustrated schematically in Figure 13. In this figure the complicated portion of the diagram is represented by a box labeled R in order to simplify the diagram and indicate that the deformation can be carried out in more general situations. The portion of the diagram contained in the box is called a *tangle*. In the first of the steps, R is unmoved while each of the vertical strands on the right is lifted up and over R and laid down to the left. In the second step the entire diagram is rotated 180 degrees about the center point of the diagram.

An unoriented link with the property that changing all the crossings yields the same link is called *amphicheiral*. (It is important here to ignore orientations, since taking the mirror image changes the linking number: unless the linking number is 0, an oriented link cannot be equivalent to its mirror image.)

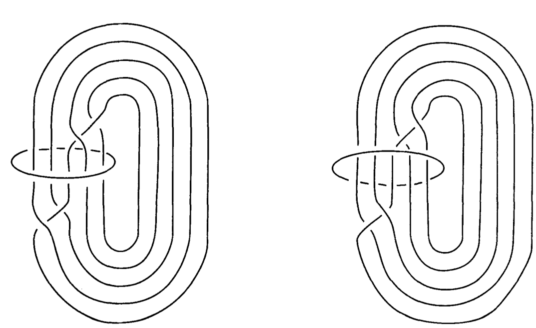


Figure 12. Amphicheiral link.

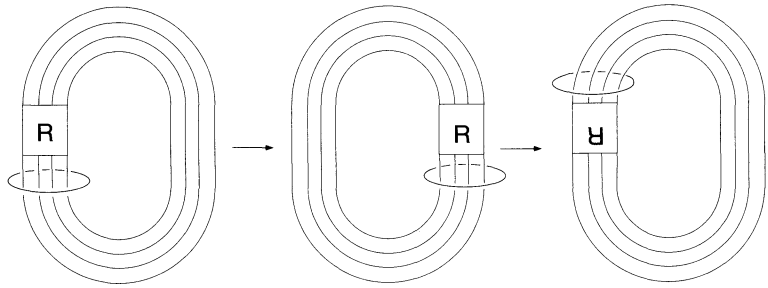


Figure 13. A two step deformation.

Other examples of amphicheiral links include the unoriented unlink and the Hopf link, depicted as U_2 and H in Figure 2.

For an example of a link that is not amphicheiral, consider the Whitehead link and its mirror image $\overline{W}$, as represented in Figure 14. If, as *unoriented* links, these were equivalent, then clearly for some choice of orientation the links would be equivalent as oriented links. For the orientations illustrated in the figure, the difference $\lambda(W) - \lambda(U_2) = -1$, whereas $\lambda(\overline{W}) - \lambda(U_2) = 1$. Hence $\lambda(\overline{W}) - \lambda(W) = 2$, so with these orientations the Whitehead link and its mirror image are not equivalent. A similar calculation applies for any other choice of orientations, showing that the Whitehead link is not amphicheiral.

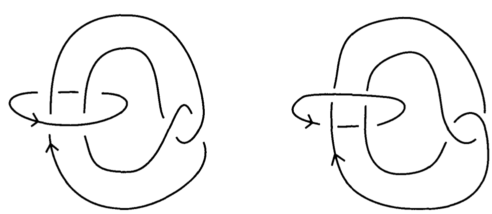


Figure 14. Right- and left-handed Whitehead links.

With the examples of the unlink, the Hopf link, and the link of Figure 12, particular links of linking number 0, 1, and 5 are seen to be amphicheiral. Given the example with linking number 5, the reader should have no trouble constructing amphicheiral links of arbitrary odd linking number. How to construct amphicheiral links of even linking number is less clear. Here is an unexpected result, the proof of which follows easily using the enhanced linking number.

Theorem 5.1. *If an unoriented link L is amphicheiral, then (for any choice of orientation) its linking number n is not congruent to* 2 *modulo* 4.

Proof. Choose an orientation for L, and let $\overline{L}$ denote the link that results by changing all the crossings of L. If L is amphicheiral, then for some choice of orientation of $\overline{L}$ the (now oriented) links L and $\overline{L}$ will be equivalent. In particular, $\lambda(L) - \lambda(\overline{L}) = 0$. Since the link L is unknown, it will be helpful to rewrite the equation $\lambda(L) - \lambda(\overline{L}) = 0$ in terms of a specific, perhaps inequivalent, link T_n:

$$(\lambda(L) - \lambda(T_n)) + (\lambda(T_n) - \lambda(\overline{T}_n)) + (\lambda(\overline{T}_n) - \lambda(\overline{L})) = 0.$$

Of course, since the relationship between L and T_n is unknown, the difference $\lambda(L) - \lambda(T_n)$ cannot be computed exactly. However, whatever sequence of crossing changes converts L to T_n translates into a sequence of crossing changes that converts $\overline{L}$ into $\overline{T}_n$; the roles of right- and left-handed crossing changes are switched. Taking into account the resulting sign changes, and other sign changes coming from the choice of orientation, yields

$$(\lambda(L) - \lambda(T_n)) = (\lambda(\overline{T}_n) - \lambda(\overline{L})).$$

Combining this with the explicit value of $\lambda(\overline{T}_n) - \lambda(T_n)$ given in equation (3) yields:

$$\lambda(L) - \lambda(T_n) = \frac{n^3 - n}{12}.$$

A small calculation using the factorization $n^3 - n = (n-1)n(n+1)$ shows that $(n^3 - n)/12$ is not an integer in the case that $n \equiv 2 \bmod (4)$, thus completing the proof. ■

The question remains as to what linking numbers can occur for amphicheiral links. All that is left to determine is the case of linking numbers congruent to 0 modulo 4. The unlink is amphicheiral of linking number 0, and the link illustrated in Figure 15 is amphicheiral with linking number 4. (Notice that in this example the deformation that converts the link into the link with all crossings changed has the effect of interchanging the two components. This is the first example of an amphicheiral link presented here for which such an interchange is required.) Similar examples can be found with linking number k^2, but beyond this no examples are presently known.

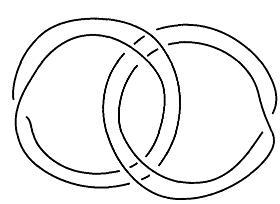


Figure 15. Amphicheiral link with linking number 4.

Open problems. Find an amphicheiral link of linking number $4k$ for each positive integer k. Also, find an amphicheiral link of linking number 4, and more generally of linking number $4k$, in which it is not necessary to switch the two components in relating L and $\overline{L}$, as is required for the link in Figure 15.

6. BRUNNIAN LINKS.

The Borromean link. That λ can be used to distinguish two-component links is to be expected, given that it is an invariant of such links. That it is effective in working with three-component links comes as a surprise. There is no better place to demonstrate this than with the Borromean link, labelled B in Figure 16. This link has always held fascination for knot theorists. If any one of the three components is removed, the unlink U_2 results. Yet, as will be shown, B itself is not trivial; that is, it is distinct from U_3, the unlink of three components. (See the article [**11**] for another approach to studying this link.) The application of λ to prove that the Borromean link is nontrivial begins with a study of the banding of links.

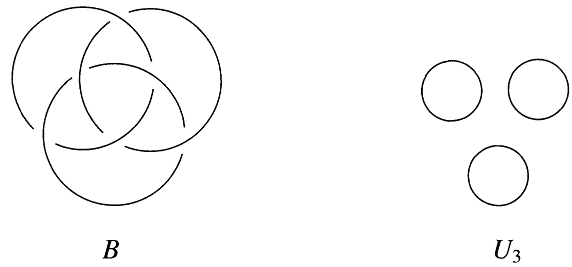


Figure 16. The Borromean link and unlink.

Starting with the unlink of three components, a two-component link can be formed by banding together two of the components, as indicated in Figure 17 (ignore the arrow in the figure for the moment). The following theorem states that the value of λ on this link is the same as its value on the unlink U_2, regardless of the choice of band. It will be seen that banding two of the components of the Borromean link together results in a link for which the value of λ differs from that of the unlink. Hence, the Borromean link cannot be trivial.

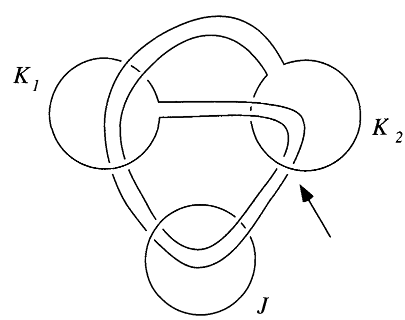


Figure 17. The banding operation.

Theorem 6.1. *If the link* (K, J) *is formed from the unlink* U_3 *by banding together two components, then* $\lambda(K, J) - \lambda(U_2) = 0$.

Proof. Suppose that the band joins components K_1 and K_2 of the unlink. In Figure 17 the band joining K_1 and K_2 passes under K_2 at the point marked with the arrow. It should be clear that if the band never passed under K_2, then the resulting link would be the trivial link U_2. In the illustrated example, if two crossing changes were made at the arrow, then where the band now passes under K_2 it would instead pass over it. In general, a series of *pairs* of crossing changes will lead to a band that does not pass under K_2.

To prove the theorem it thus suffices to show that each pair of crossing changes has no effect on the value of λ. The crossing changes are of opposite sign, so it remains to check only that the value of $\mathrm{lk}(\rangle, J)\mathrm{lk}(\langle, J)$ in the crossing change formula is the same for each pair. Each of the terms in this product can be identified as follows. If the band is cut (at the point of the arrow in the illustrated example), it splits into two pieces. In Figure 18 the cut has been made, and the band has split into two long "fingers," one extending from K_1 and one from K_2. A careful calculation shows that in the crossing change formula the two terms of the product are given as (plus and minus) the number of times the finger extending from K_2 "passes through" J. (To be more precise, the number of times the band passes through J is computed as follows. The band itself is first given an orientation. This orients each of the two fingers. One then computes the number of times the finger from K_2 passes under J moving from right to left, minus the number of times this finger passes under J moving from left to right.) This is independent of which of the two crossings are smoothed. The result follows since the two crossing changes are of opposite sign. ■

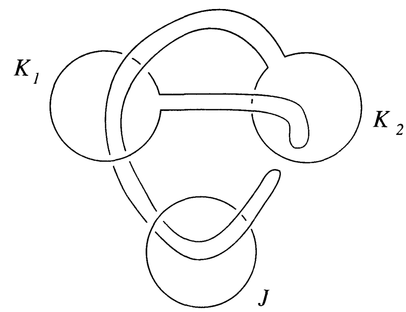


Figure 18. The banding operation.

In Figure 19 a two-component link formed from the Borromean link by banding together two of the components is shown. A single crossing change results in the unlink, and an application of the crossing change formula determines that the value of λ on this link differs from that of the unlink by -1. By Theorem 6.1, if the Borromean link were trivial, then this difference would not occur. Accordingly, the Borromean link is nontrivial.

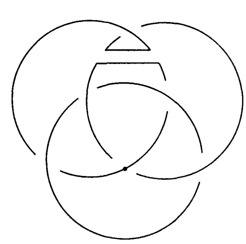


Figure 19. Banding the Borromean link.

More Brunnian examples. The Borromean link is an example of a *Brunnian* link, named after the nineteenth-century mathematician H. Brunn who first studied generalizations of the Borromean link. Brunnian links are links that become unlinked when any one component is removed. In general, distinguishing Brunnian links is difficult: linking numbers will not suffice, and considering sublinks is also unhelpful. Here we expand on the example of the Borromean link. The proof of the following theorem mimics the proof of Theorem 6.1.

Theorem 6.2. *If a two-component link is formed from a three-component Brunnian link* (J_1, J_2, J_3) *by banding together two components, the value of* λ *for the link is independent of the choice of band joining those two components.*

Consider now the Brunnian link B_2 on the left in Figure 20.

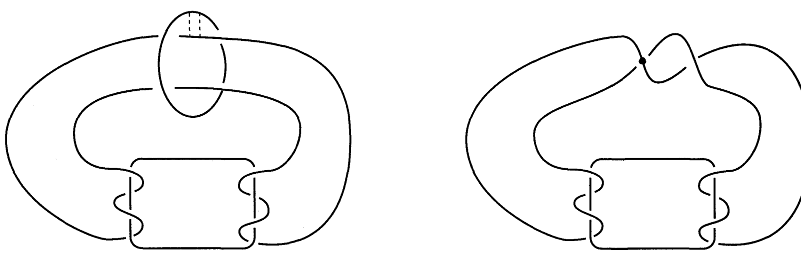


Figure 20. Brunnian link B_2, before and after banding.

Banding two of the components together along the dotted band in the figure results in the link drawn on the right of Figure 20. The value of λ on this link differs from that of $\lambda(U_2)$ by 4. (Change the crossing indicated by the dot to get to the unlink.) It takes a little more manipulation to see that banding together any of the other pairs of components also yields a two-component link with λ-value differing from $\lambda(U_2)$ by 4. In the case of the Borromean link, after banding the difference was -1. Consequently, B_2 is different from both U_2 and B.

Exercise 6.3. Describe Brunnian links B_n so that, whichever two components are banded together to form a link B'_n, one has $\lambda(B'_n) - \lambda(U_2) = n^2$, and hence construct an infinite family of distinct Brunnian links.

7. PERIODICITY. Consider the two knots pictured in Figure 21. The one on the left is said to have period five since if the diagram is rotated through an angle of $2\pi/5$ about the origin, marked with the black dot, it is unchanged. Similarly the diagram on the right represents a knot of period three. The sort of question of interest here is whether it is possible to deform the first knot so that it reveals a period three periodicity, where the deformation should never cross the black dot.

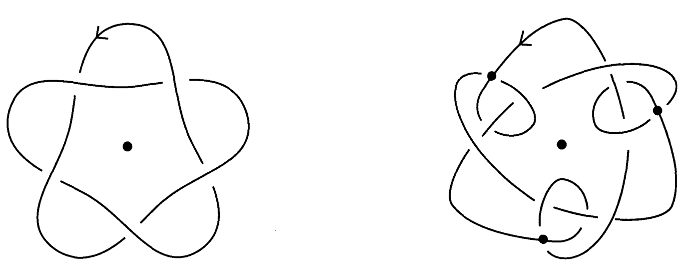


Figure 21. Periodic knots.

Such problems can be placed in the context of linking by broadening the notion of link. If the plane of the paper is viewed as the (x, y)-plane in R^3, then the z-axis, denoted Z, would be perpendicular to the plane of the paper, appearing as a central point in the diagram. The notion of link can be expanded to include pairs (K, Z), where K is a knot disjoint from the z-axis. The diagram of such a link consists of a diagram for K along with the origin in the plane, the projection of Z, indicated by a large dot.

The theory of linking numbers and enhanced linking numbers goes through as before, though now only crossing changes in K, not in Z, are considered. The linking number $\text{lk}(K, Z)$ is computed by counting the number of times the projection of K crosses the *positive* x-axis, (or any other ray emanating from the origin) counting 1 if it is moving counterclockwise and -1 if it is moving clockwise. Both of the previous examples have linking number 2.

Definition 7.1. A link (K, Z) is called *periodic of period p* if K can be deformed in the complement of the z-axis into a knot that is transformed to itself by a rotation of R^3 about the z-axis through an angle $2\pi/p$.

Theorem 7.2. *If links (K_1, Z) and (K_2, Z) both have the same linking number and are of period p, then the difference $\lambda(K_1, Z) - \lambda(K_2, Z)$ is divisible by p.*

Proof. If (K, Z) is of period p, there is a *quotient* link $(\tilde{K}, Z)$, with the same linking number. Figure 22 illustrates the formation of a quotient knot from the periodic diagram of a knot, working with the diagram on the right in Figure 21. The quotients of the links in Figure 21, under the period five and period three symmetries are then illustrated in Figure 23.

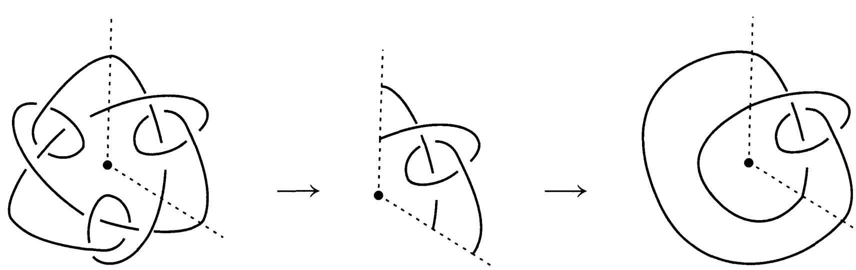


Figure 22. Building a quotient knot.

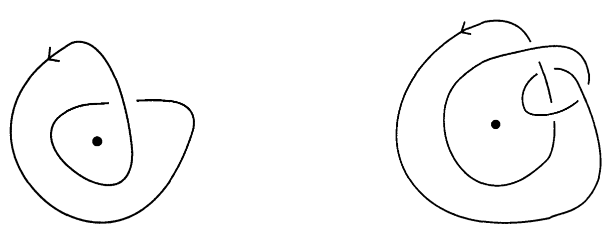


Figure 23. Quotient knots.

Now, if (K_1, Z) and (K_2, Z) have the same linking number, then consider the quotients $(\tilde{K}_1, Z)$ and $(\tilde{K}_2, Z)$. Since they have the same linking number, a series of crossing changes in $\tilde{K}_1$ will change it into $\tilde{K}_2$. Each of these crossing changes determines p crossing changes in K_1. Hence, the total set of crossing changes that converts K_1 into K_2 will split into sets of p crossing changes, where the elements of each set lead to the same change in the value of λ. Thus, the total change in the value of λ will be divisible by p. ■

Example. The reader should check that the difference of λ for the two links (K_1, Z) and (K_2, Z) arising from Figure 21 is 1. (Three crossing changes at the points marked with dots are needed to eliminate the kinks in L_2, yielding a link $L_2' = (K_2', Z)$; these don't change λ. A crossing change in L_1 converts it into L_2'.) Since 1 is divisible by neither 3 nor by 5, the first link in the figure is not of period 3 and the second is not of period 5.

This represents the application of the difference formula for λ to shed light on a particular pair of links. Here is a much deeper application.

Theorem 7.3. *If a link (K, Z) has linking number 2, then it has at most a finite number of periods.*

Proof. The period 5 link in Figure 21 will henceforth be denoted L_5. There is an analog L_k of period k and linking number 2 whenever k is odd.

If (K, Z) has period n, then it has period p^e for each prime power divisor p^e of n. Thus it suffices to show that (K, Z) has at most finitely many periods that are prime powers. Suppose that (K, Z) has period p^e for some prime p and positive integer e. Suppose that it also has period q^f, where q is a prime not necessarily distinct from p. By Theorem 7.2 there are integers a and b for which the following equations hold:

$$\lambda(K, Z) - \lambda(L_{p^e}) = ap^e, \qquad \lambda(K, Z) - \lambda(L_{q^f}) = bq^f.$$

Subtracting the equation on the left from the equation on the right yields

$$\lambda(L_{p^e}) - \lambda(L_{q^f}) = bq^f - ap^e.$$

If $p^e > q^f$, then L_{p^e} can be converted into L_{q^f} using $(p^e - q^f)/2$ crossing changes. Hence $\lambda(L_{p^e}) - \lambda(L_{q^f}) = (p^e - q^f)/2$. In the case that $q^f > p^e$ identical reasoning shows that $\lambda(L_{q^f}) - \lambda(L_{p^e}) = (q^f - p^e)/2$. In either case we have

$$\frac{p^e - q^f}{2} = bq^f - ap^e$$

or

$$(1 + 2a)p^e = (1 + 2b)q^f. \tag{4}$$

Suppose first that there is an infinite set of primes q_i with powers among the periods of (K, Z). Then there would be an infinite set of equalities (4): $(1 + 2a)p^e = (1 + 2b_i)(q_i)^{f_i}$. This implies that $1 + 2a$ is divisible by an infinite set of primes, which is impossible since it is odd and thus nonzero.

Accordingly, at most finitely many primes have powers that turn up among the periods of (K, Z). Suppose that for some prime p arbitrarily large powers of p, say p^{e_i}, appear among the periods. Let one of those periods be denoted simply p^e. An infinite set of equalities $(1 + 2a)p^e = (1 + 2b_i)p^{e_i}$ with e_i arbitrarily large would arise. But this would imply that $1 + 2a$ is divisible by arbitrarily large powers of p, which is again impossible because it is nonzero. ∎

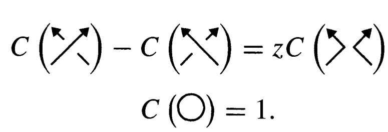
Note. An argument similar to the proof of Theorem 7.3 can be carried out for an arbitrary linking number n. There are a fair number of delicate issues that arise, but the following counterpart of (4) surfaces:

$$\left(\frac{n^3 - n}{6} + 2a\right) p^e = \left(\frac{n^3 - n}{6} + 2b\right) q^f.$$

Since $(n^3 - n)/6$ is odd if $n \equiv 2 \bmod(4)$, an argument like the earlier one shows that for these linking numbers there are no links with an infinite number of periods. (For $n = 1$ there is a link with an infinite number of periods, but for all other linking numbers this situation cannot occur. We do not know a purely algebraic proof of this fact analogous to the one just given for $n = 2$.)

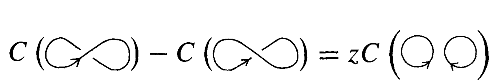
8. DEFINING λ: THE CONWAY POLYNOMIAL. There is an obvious gap in this article up to this point, namely, the proof of Theorem 4.1. Basically it has been seen that, if there is an invariant λ satisfying the crossing change formula given in Definition 4.2, then the subsequent results of the article hold. We have yet to establish the existence of such a λ. This gap will now be filled. The definition of λ depends on the use of knot polynomials—in particular, the Conway polynomial—so the discussion begins with this invariant. It should be mentioned that the original definition of λ given in [**8**] was based not on polynomial invariants, but rather on a subtle invariant of 3-manifolds called the Casson-Walker invariant. The connection with polynomials developed from the work of Kanenobu, Miyzawa, and Tani [**5**].

The Alexander and Conway polynomials. In 1928 Alexander showed how to associate with each diagram of a knot or link a polynomial. Unfortunately, Reidemeister moves change the resulting polynomial, so it is not an invariant of links. However, Alexander did prove that any two polynomials associated with the same knot or link would differ by multiplication by a power t, the name used for the variable in the Alexander polynomial. The proof is outlined in [**10**]. This polynomial is now referred to as the *Alexander polynomial.* In 1975 Conway [**4**] observed that by properly normalizing the Alexander polynomial of a link L one obtains a polynomial invariant of knots and links, now called the *Conway polynomial* and denoted $C(L)(z)$. Here L represents a knot or link and $C(L)$ is a polynomial in a variable named z. (There is a choice of sign in defining the Conway polynomial and references are not all consistent.) By Alexander's work the Conway polynomial is well-defined. One of the main benefits of Conway's approach was that it presented a crossing change formula for the polynomial. Conway observed that this crossing change formula and the fact that the Conway polynomial of the unknot is 1 completely determine the polynomial for all links. Conway also noted that the crossing change formula gives a simple technique for computing the polynomial, much simpler than Alexander's determinant formula. Here are the essential formulas:


(5)

Recall that the link represented as  in (5) is called the *smoothed* link.

Sample computations of the Conway polynomial. A basic application of the crossing formula is the following:



Since the two knots appearing on the left side are the same (the unknot), it follows that the Conway polynomial of the unlink is 0. Similar computations give:

Theorem 8.1. *$C(U_n) = 0$ for all $n \geq 2$.*

As a more complicated example, consider the knot drawn in Figure 24. This knot is usually denoted 4_2 in tables of knots and is called the *figure eight knot*.

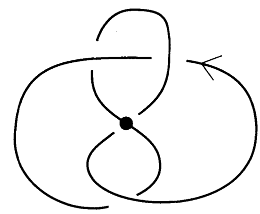
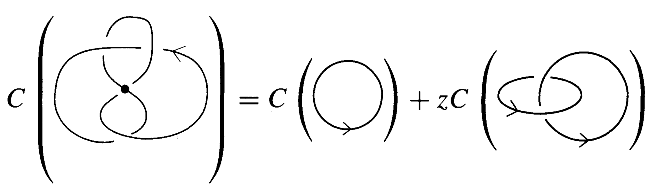


Figure 24. The knot 4_2.

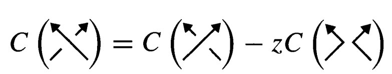
Applying Conway's formula (5), rewritten as

yields the following:

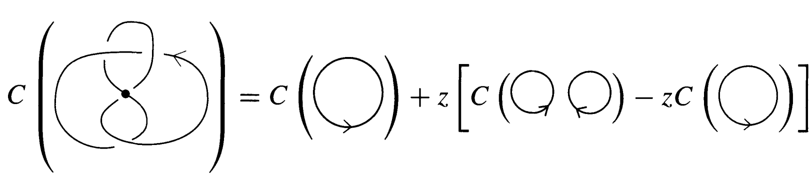


(6)

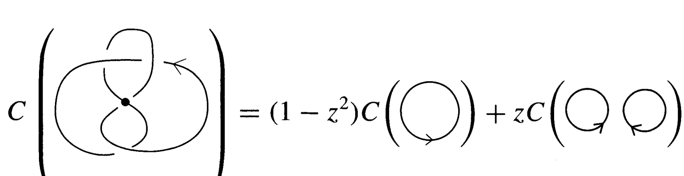
The reader is encouraged to check that making the crossing change and the smoothing at the crossing marked with the dot results, after deformation, in the knot and link that appear on the right-hand side of (6). We again apply Conway's formula (5), written now as



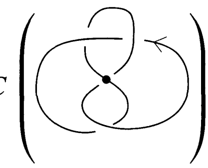
to the Hopf link that appears in (6), working at either crossing point, to find that



Expanding this equation leads to



On the basis of Theorem 8.1 we finally arrive at

C  $= 1 - z^2.$

Basic properties of the Conway polynomial. The individual coefficients of the Conway polynomial are link invariants in their own right. These coefficients, viewed as invariants $c_i(L)$ $(i \geq 0)$ for a link L, are defined by the expansion

$$C(L)(z) = \sum c_i(L) z^i.$$

Notice that smoothing a crossing changes the number of components of a link by one and that multiplication by z switches odd and even polynomials. (Here, by an even polynomial we mean a polynomial that represents an even function, a polynomial of the form $\sum_{i=0}^{n} a_{2i} z^{2i}$; likewise for odd polynomials.) It then follows quickly from induction that a link with an even (respectively, odd) number of components has an odd (respectively, even) Conway polynomial.

The Conway polynomial of a link can be computed by repeatedly smoothing and changing crossings until unlinks are achieved, as was done in the example of the figure eight knot. We have already seen that the Conway polynomial of an unlink with more than one component is 0, so contributions to the Conway polynomial arise only when the unlink has exactly one component, that is, when it is an unknot. It takes at least $n - 1$ smoothings to reduce an n-component link to an unknot. The following theorem then results.

Theorem 8.2. *For a link L of n components, $c_i(L) = 0$ for $i < n - 1$.*

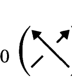
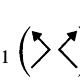
Here are three basic results concerning the first nontrivial coefficient of the Conway polynomial of links of one, two, and three components.

Theorem 8.3. *For a one-component link L, $c_0(L) = 1$.*

Theorem 8.4. *For a two-component link L, $c_1(L) = \mathrm{lk}(L)$.*

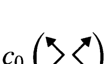
Theorem 8.5. *For a three-component link L with pairwise linking numbers α, β, and γ, $c_2(L) = \alpha\beta + \beta\gamma + \alpha\gamma$.*

Proof. In the case of a knot, one obtains from the crossing change formula the relation

c_0  $- c_0$  $= c_{-1}$  $= 0.$

Hence, the value of c_0 is independent of the knot, and since for the unknot $c_0 = 1$, this is true for all knots.

In the case of a link with two components, one has

c_1  $- c_1$  $= c_0$  .

If the crossing change is between different components, then the smoothed link is a knot and the difference is 1. If the crossing change is between a component and itself, the smoothed link has three components and the difference is 0. As a result, changes in $c_1(L)$ exactly parallel changes in the linking number. Furthermore, $c_1(L)$ agrees with the linking number for the unlink, and hence must be the same as the linking number.

The third theorem is the most technical to prove, and the proof won't be presented in detail here, although the ideas involved are not especially deep. One needs to check only that the two expressions $c_2(L)$ and $\alpha\beta + \beta\gamma + \alpha\gamma$ agree on the unlink of three components (both are 0) and that both behave the same way under crossing changes, which is a consequence of the Conway crossing change formula.

The definition of $\boldsymbol{\lambda}$. It is now time to demonstrate that there really is a link invariant that satisfies the crossing change formula given for λ in Definition 4.2. Here is the definition.

Definition 8.6. For a two-component link $L = (K, J)$, define

$$\lambda(L) = c_3(L) - c_1(L)(c_2(K) + c_2(J)). \tag{7}$$

Since the c_i are integral invariants of L, so is λ.

Theorem 8.7. *The invariant λ of two-component links defined by (7) satisfies the crossing change formula given by (1) in Definition 4.2.*

Proof. The goal is to show that

$$\lambda\left(\times_+, J\right) - \lambda\left(\times_-, J\right) = \mathrm{lk}\left(\rangle, J\right)\mathrm{lk}\left(\langle, J\right) \tag{8}$$

Suppose that the link $L = (K, J)$ has linking number n. Then equation (7) can be rewritten as $\lambda(L) = c_3(L) - n(c_2(K) + c_2(J))$. Only crossing changes in K are considered; working with J is identical. Rewriting the difference in terms of the definition of λ changes (8) into the following:

$$c_3\left(\times_+, J\right) - c_3\left(\times_-, J\right) - n\left(c_2\left(\times_+\right) - c_2\left(\times_-\right)\right) = \mathrm{lk}\left(\rangle, J\right)\mathrm{lk}\left(\langle, J\right).$$

Applying the defining relation (5) for the Conway polynomial ($C(\times_+) - C(\times_-) = zC(\rangle\langle)$) turns this into

$$c_2\left(\rangle\langle, J\right) - nc_1\left(\rangle\langle\right) = \mathrm{lk}\left(\rangle, J\right)\mathrm{lk}\left(\langle, J\right). \tag{9}$$

Since

$$\mathrm{lk}\left(\rangle, J\right) + \mathrm{lk}\left(\langle, J\right) = n$$

and

$$c_1\left(\rangle\langle\right) = \mathrm{lk}\left(\rangle, \langle\right),$$

(9) becomes

$$c_2\left(\rangle\langle, J\right) - \left(\mathrm{lk}\left(\rangle, J\right) + \mathrm{lk}\left(\langle, J\right)\right)\mathrm{lk}\left(\rangle\langle\right) = \mathrm{lk}\left(\rangle, J\right)\mathrm{lk}\left(\langle, J\right).$$

Notice that $(\rangle\langle, J)$ represents a three-component link: the first component has been smoothed. Theorem 8.5 can thus be applied to expand the c_2 term, and this quickly shows that this final equation holds and the proof is complete. ■

The value of λ. At this point a formal definition of λ has been given, along with a crossing change formula. The reader might have observed that the value of λ has not as yet been computed for any link. Clearly $\lambda(U_2) = 0$. This section concludes with a more interesting family of examples.

Theorem 8.8. *For the link T_n, $\lambda(T_n) = 0$.*

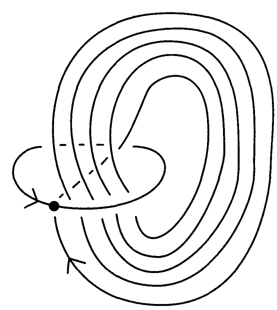


Figure 25. T_5.

Proof. Since both components of T_n are unknotted, it follows that $\lambda(T_n) = c_3(T_n)$. A crossing change at the crossing marked with a dot in Figure 25 (which shows T_n for $n = 5$) converts T_n into T_{n-1}. Smoothing the crossing results in an unknot, denoted U_1. Hence

$$c_3(T_n) - c_3(T_{n-1}) = c_2(U_1) = 0,$$

i.e., $c_3(T_n) = c_3(T_{n-1})$. This reduces the calculation to that for T_0. But T_0 is the unlink U_2, for which $c_3(U_2) = 0$. (It was shown earlier that, in fact, $C(U_2) = 0$.) ■

9. FINITE TYPE INVARIANTS. We cannot conclude this article without mentioning the connection of λ to one of the most active areas in knot theory in recent years, the theory of *finite type invariants*. A more formal introduction to finite type invariants is contained in the paper [**3**]. The crossing change formulas that have been considered here focus on what happens to an integer-valued knot invariant, say μ, if a single crossing is changed. One can instead consider what happens when the focus changes to multiple crossings. For instance, suppose that two crossings in a knot or link diagram are selected. The following sum then arises naturally:

$$\mu\left(\times\times\right) - \mu\left(\times\times\right) - \mu\left(\times\times\right) + \mu\left(\times\times\right).$$

Similarly, if one focuses on k crossings in a diagram there is a corresponding sum of 2^k terms, where the signs are determined by the parity of the number of left-handed crossings in each of the individual summands. This sum is called the *type k difference* for μ.

Definition 9.1. The link invariant μ is said to be of *finite type k* ($k = 0, 1, 2, \ldots$) if for every link L the type $k+1$ difference for μ is 0 for all possible choices of $k+1$ crossings in all possible diagrams for L.

The following is a somewhat tricky exercise, but one with a simple solution once found.

Exercise 9.2. The Conway coefficient c_k is a type k finite type invariant.

Two link invariants μ_1 and μ_2 are added in the obvious way: $(\mu_1 + \mu_2)(L) = \mu_1(L) + \mu_2(L)$. With addition so defined, the family $\mathcal{F}$ of all link invariants forms an Abelian group. The set $\mathcal{F}_k$ of type k invariants constitutes a subgroup of $\mathcal{F}$. It follows from the definition that $\mathcal{F}_k$ is contained in $\mathcal{F}_{k+1}$ for all $k \geq 0$. A basic result in the subject is the following:

Theorem 9.3. *The quotient $\mathcal{F}_{k+1}/\mathcal{F}_k$ is a finitely generated free Abelian group.*

An outstanding problem in link theory is to determine the rank of these quotient groups. It is known that the rank grows quickly as a function of k. If one restricts attention to two-component links of a given linking number, say n, then in the difference formula one must restrict the type k difference equation so that each individual crossing represents the crossing of one component with itself. This ensures that each link in the equation has the same linking number. With this modification one can refer to *type k invariants of links with linking number n*. For any given linking number, one has the following:

Theorem 9.4. *For all n, λ is a type 1 invariant of two-component links with linking number n.*

The proof Theorem 9.4 is a basic application of the crossing change formula for λ and the fact that changing a self-crossing in a link does not change the linking number. A much deeper result is:

Theorem 9.5. *Every type 1 invariant of two-component links with linking number n is of the form $a\lambda + b$ for some constants a and b.*

In other words, for each n, λ stands out as essentially the unique type 1 invariant of two-component links with linking number n.

Little is understood concerning finite type invariants for disjoint links of type greater than 1. It is assumed that this group is infinitely generated if the type is greater than 1, but a proof of this has yet to be found.

ACKNOWLEDGMENTS. I would like to thank Paul Kirk for working with me in developing the enhanced linking number. The suggestions and corrections given to me by Scott Baldridge, Jae Choon Cha, Charles Delman, Hyungsoo Do, Lynn Greenfield, Ben Himpel, Eric Livingston, John McAtee, Kevin Pilgrim, and Chris Tuffley were of great help. I would also like to thank the referee for comments that improved the exposition.

REFERENCES

1. J. Alexander, Topological invariants of knots and links, *Trans. Amer. Math. Soc.* **30** (1928) 275–306.
2. P. Andersson, The color invariant for knots and links, this MONTHLY **102** (1995) 442–448.
3. J. Birman and X-S. Lin, Knot polynomials and Vassiliev's invariants, *Invent. Math.* **111** (1993) 225–270.
4. J. H. Conway, An enumeration of knots and links, and some of their algebraic properties, in *Computational Problems in Abstract Algebra*, Pergamon, Oxford, 1970, pp. 329–358.
5. T. Kanenobu, Y. Miyazawa, and A. Tani, Vassiliev link invariants of order three, *J. Knot Theory Ramifications* **7** (1998) 433–462.
6. L. Kauffman, New invariants in the theory of knots, this MONTHLY **95** (1988) 195–242.
7. ———, *On Knots*, Annals of Mathematics Studies, no. 115, Princeton University Press, Princeton, 1987.
8. P. Kirk and C. Livingston, Vassiliev invariants of two component links and the Casson-Walker invariant, *Topology* **36** (1997) 1333–1353.
9. W. B. R. Lickorish and K. Millett, The new polynomial invariants of knots and links, *Math. Mag.* **61** (1988) 3–23.
10. C. Livingston, *Knot Theory*, Carus Mathematical Monographs, no. 24, Mathematical Association of America, Washington, D.C., 1993.
11. O. Nanyes, An elementary proof that the Borromean rings are non–splittable, this MONTHLY **100** (1993) 786–789.
12. N. Sato, Cobordisms of semiboundary links, *Topology Appl.* **18** (1984) 225–234.
13. M. Sullivan, Knot factoring, this MONTHLY **107** (2000) 297–315.

CHARLES LIVINGSTON studied mathematics as an undergraduate at UCLA and MIT; his graduate study was done under the direction of Rob Kirby at the University of California, Berkeley, where he received his Ph.D. in 1980. Following an instructorship at Rice University he moved to his present position on the mathematics faculty at Indiana University. In his first effort to bring an appreciation of knot theory to a wider audience, Livingston wrote the book *Knot Theory* for the MAA Carus Series in 1993. In addition to studying knots and links, Livingston has worked at improving the education of journalists in mathematics and statistics; in joint work with Paul Voakes, a colleague in the IU School of Journalism, he is writing a book for practicing journalists and journalism students.
Department of Mathematics, Indiana University, Bloomington, IN 47405
livingst@indiana.edu