Absolutely Abnormal Numbers

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1. INTRODUCTION. A normal number is one whose decimal expansion (or expansion to some base other than 10) contains all possible finite configurations of digits with roughly their expected frequencies. More formally, when $b \geq 2$ is an integer, let

$$N(\alpha; b, a, x) = \#\{1 \leq n \leq x\colon \text{the } n\text{th digit in the base-}b \text{ expansion of } \alpha \text{ is } a\} \quad (1)$$

denote the counting function of the occurrences of the digit a $(0 \leq a < b)$ in the b-ary expansion of the real number α, and define the corresponding limiting frequency

$$\delta(\alpha; b, a) = \lim_{x\to\infty} x^{-1} N(\alpha; b, a, x), \quad (2)$$

if the limit exists.

The number α is *simply normal* to the base b if the limit defining $\delta(\alpha; b, a)$ exists and equals $1/b$ for each integer $a \in [0, b)$. When α is a b-adic fraction a/b^n, which has one b-ary expansion with all but finitely many digits equaling zero and another b-ary expansion with all but finitely many digits equaling $b-1$, these limiting frequencies are not uniquely defined; however, such an α is not simply normal to the base b in either case.

A number is *normal* to the base b if it is simply normal to each of the bases $b, b^2, b^3, \ldots$. This is equivalent to demanding that for any finite string $a_1 a_2 \ldots a_k$ of base-b digits, the limiting frequency of occurrences of this string in the b-ary expansion of α (analogous to (2)) exists and equals $1/b^k$; see [**6**, Chapter 8].

Champernowne showed [**1**] that the number $0.12345678910111213\ldots$ formed by concatenating all of the positive integers together into a single decimal is normal to base 10 (an analogous construction works for any base $b \geq 2$), and this sort of example has been generalized; see [**2**] or [**3**]. It is known that almost all real numbers are normal to any given base b [**6**, Theorem 8.11], and consequently almost all real numbers are *absolutely normal*, i.e., normal to all bases $b \geq 2$ simultaneously. On the other hand, no one has proved a single naturally occurring real number to be absolutely normal.

Let us call a number *abnormal* to the base b if it is not normal to the base b, and *absolutely abnormal* if it is abnormal to all bases $b \geq 2$ simultaneously. For example, every rational number r is absolutely abnormal: any b-ary expansion of r eventually repeats, say with period k, in which case r is about as far from being simply normal to the base b^k as it can be. Even though the set of absolutely abnormal numbers is the intersection of countably many sets of measure zero, Maxfield has pointed out that the set of absolutely abnormal numbers is uncountable and dense in the real line [**5**]; later, Schmidt gave a complicated constructive proof of this fact [**7**]. In this paper we exhibit a simple construction of a specific irrational (in fact, transcendental) real number that is absolutely abnormal. In fact, our construction easily generalizes to a construction giving uncountably many absolutely abnormal numbers in any open interval.

It is instructive to consider why constructing an irrational, absolutely abnormal number is even difficult. Since we already know that rational numbers are absolutely abnormal, our first thought might be to choose an irrational number whose b-ary expansions mimic those of rational numbers for long stretches, i.e., an irrational number with very good rational approximations. Thus a natural class to consider is the *Liouville numbers*, those real numbers β such that for every positive integer m, there exists a rational number p/q (not necessarily in lowest terms) satisfying

$$0 < \left|\beta - \frac{p}{q}\right| < \frac{1}{q^m}. \tag{3}$$

These Liouville numbers are all transcendental (see Lemma 6)—in fact Liouville introduced these numbers precisely to exhibit specific transcendental numbers, and the oft-cited example

$$\beta = \sum_{n=1}^{\infty} 10^{-n!} = 0.110001000000000000000001 00\dots$$

is usually the first provably transcendental number that a student sees.

Clearly β is abnormal to the base 10. How would we go about showing, for example, that β is abnormal to the base 2? We could try to argue that the binary expansion of β agrees with that of each of the rational numbers

$$\beta_k = \sum_{n=1}^{k} 10^{-n!} \tag{4}$$

through about the $((n+1)!\log_2 10)$-th binary digit. Since each β_k is rational and thus abnormal to the base 2, can we conclude that β itself is abnormal to the base 2?

Not quite: it seems that we would have to show that there is a fixed power 2^n such that infinitely many of the β_k were not simply normal to the base 2^n. (For each β_k there is *some* power 2^{n_k} such that β_k is not simply normal to the base 2^{n_k}, but these exponents n_k might very well grow with k.) In fact, it is not hard to show (using the fact that 2 is a primitive root modulo every power of 5) that any 10-adic fraction that is not a 2-adic fraction—including each β_k—*is* simply normal to the base 2! In general, without actually computing binary expansions of specific fractions, it seems impossible to rule out the incredible possibility that the β_k are accidentally simply normal to bases that are high powers of 2. In summary, any Liouville number we write down seems rather likely to be absolutely abnormal, but actually proving its absolute abnormality is another matter.

To circumvent this difficulty, we construct a Liouville number whose successive rational approximations are b-adic fractions with b varying, rather than all being 10-adic fractions as in (4). The existence of such Liouville numbers can certainly be proved using just the fact that the b-adic fractions are dense in the real line for any integer $b \ge 2$; however, our construction is completely explicit. We first give the complete construction of our irrational, absolutely abnormal number and then show afterwards that it has the required properties.

2. THE CONSTRUCTION AND PROOF. We begin by defining a sequence of integers

$$d_2 = 2^2, \quad d_3 = 3^2, \quad d_4 = 4^3, \quad d_5 = 5^{16}, \quad d_6 = 6^{30,517,578,125}, \dots$$

with the recursive rule

$$d_j = j^{d_{j-1}/(j-1)} \quad (j \ge 3). \tag{5}$$

This sequence exhibits the pattern

$$d_4 = 4^{3^{2-1}}, \quad d_5 = 5^{4^{\left(3^{2-1}-1\right)}}, \quad d_6 = 6^{5^{\left(4^{\left(3^{2-1}-1\right)}-1\right)}}, \ldots$$

which in general gives the typesetting nightmare

$$d_j = j^{(j-1)^{\left((j-2)^{\left((j-3)^{\left(\cdots^{\left(4^{\left(3^{2-1}\right)}-1\right)}\cdots\right)}-1\right)}-1\right)}}. \tag{6}$$

Using these integers, we define the sequence of rational numbers

$$\alpha_k = \prod_{j=2}^{k} \left(1 - \frac{1}{d_j}\right), \tag{7}$$

so that $\alpha_2 = 1/4$, $\alpha_3 = 2/3$, $\alpha_4 = 21/32$, $\alpha_5 = 100{,}135{,}803{,}222/152{,}587{,}890{,}625$, and so on.

We now nominate

$$\alpha = \lim_{k\to\infty} \alpha_k = \prod_{j=2}^{\infty} \left(1 - \frac{1}{d_j}\right) \tag{8}$$

as our candidate for an irrational, absolutely abnormal number. The first few digits in the decimal expansion of α are

$$\alpha = 0.65624999999569919\underbrace{99999\ldots99999}_{23{,}747{,}291{,}559 \text{ 9s}}8528404201690728\ldots, \tag{9}$$

from which we can get an inkling of the extreme abnormality of α (at least to the base 10). We need to prove three things about α: first, that the infinite product (8) actually converges; second, that α is irrational; and finally, that α is absolutely abnormal.

It is apparent from the expressions (5) and (6) that the d_j grow (ridiculously) rapidly and hence that the infinite product (8) should indeed converge. The following lemma provides a crude inequality relating the integers d_j that we can use to prove this assertion rigorously.

Lemma 1. *$d_j > 2d_{j-1}^2$ for all $j \geq 5$.*

Proof. We proceed by induction, the case $j = 5$ being true by inspection. For $j > 5$ we surely have

$$d_j = j^{d_{j-1}/(j-1)} > 5^{d_{j-1}/(j-1)}.$$

Notice that (5) ensures that

$$\frac{d_{j-1}}{j-1} = (j-1)^{\frac{d_{j-2}}{j-2}-1} > (j-1)^{\frac{d_{j-2}}{2(j-2)}} = \sqrt{d_{j-1}},$$

and therefore

$$d_j > 5^{\sqrt{d_{j-1}}}.$$

Now using the fact that $5^x \geq x^5$ for $x \geq 5$, we conclude that

$$d_j > \left(\sqrt{d_{j-1}}\right)^5 > 2d_{j-1}^2,$$

as desired. ■

Equipped with this inequality, we can now show that the infinite product (8) converges. Moreover, we can show that α is well approximated by the rational numbers α_k. Notice that α_4 is exactly 0.65625 and α_5 is exactly 0.6562499999956992—cf. the decimal expansion (9) of α itself.

Lemma 2. *The product (8) defining α converges. Moreover, for $k \geq 2$ we have*

$$\alpha_k > \alpha > \alpha_k - \frac{2}{d_{k+1}}. \tag{10}$$

Proof. To show that the product (8) converges, we must show that the corresponding sum $\sum_{j=2}^{\infty} 1/d_j$ converges. But Lemma 1 ensures that $d_j > 2d_{j-1}$ for $j \geq 5$, and therefore

$$\sum_{j=2}^{\infty} \frac{1}{d_j} \leq \frac{1}{d_2} + \frac{1}{d_3} + \sum_{j=4}^{\infty} \frac{1}{2^{j-4}d_4} = \frac{1}{d_2} + \frac{1}{d_3} + \frac{2}{d_4} < \infty.$$

Similarly, using the fact that $1 \geq \prod(1 - x_j) \geq 1 - \sum x_j$ whenever $0 \leq x_j \leq 1$, we see that for $k \geq 3$

$$\begin{aligned}\alpha_k > \alpha = \alpha_k \prod_{j=k+1}^{\infty} \left(1 - \frac{1}{d_j}\right) &\geq \alpha_k \left(1 - \sum_{j=k+1}^{\infty} \frac{1}{d_j}\right) > \alpha_k \left(1 - \sum_{j=k+1}^{\infty} \frac{1}{2^{j-k-1}d_{k+1}}\right) \\ = \alpha_k \left(1 - \frac{2}{d_{k+1}}\right) &> \alpha_k - \frac{2}{d_{k+1}}.\end{aligned}$$

The inequalities (10) for $k = 2$ follow from those for $k = 3$, since $\alpha_2 > \alpha_3 > \alpha_3 - 2/d_4 > \alpha_2 - 2/d_3$. ■

It turns out that both the proof that α is irrational and the proof that α is absolutely abnormal hinge on the fact that each rational approximation α_k is in fact a k-adic fraction—that is, when α_k is expressed in lowest terms, its denominator divides a power of k. In other words, each time we multiply α_{k-1} by $1 - 1/d_k = (d_k - 1)/d_k$ to obtain α_k, the numerator of the latter fraction completely cancels out the denominator of α_{k-1}, so that all that remains in the denominator of α_k are the powers of k present in d_k. Proving that this always happens is an exercise in elementary number theory, which we present in the next three lemmas.

Lemma 3. *Let k and r be positive integers, and let p be a prime. If k is divisible by p^r, then $(k+1)^p - 1$ is divisible by p^{r+1}.*

Proof. Writing $k = p^r n$, the binomial theorem gives

$$(k+1)^p - 1 = (p^r n + 1)^p - 1$$

$$= \left\{(p^r n)^p + \binom{p}{p-1}(p^r n)^{p-1} + \cdots + \binom{p}{2}(p^r n)^2 + \binom{p}{1}p^r n + 1\right\} - 1$$
$$= p^{rp}n^p + \binom{p}{p-1}p^{r(p-1)}n^{p-1} + \cdots + \binom{p}{2}p^{2r}n^2 + p \cdot p^r n.$$

Since all binomial coefficients are integers, each term in this last sum is divisible by p^{r+1}. ■

Lemma 4. *For any positive integers k and m, $(k+1)^{k^m} - 1$ is divisible by k^{m+1}.*

Proof. If p^r is any prime power dividing k, an rm-fold application of Lemma 3 shows us that $(k+1)^{p^{rm}} - 1$ is divisible by p^{r+rm}. Then, since

$$(k+1)^{k^m} - 1 = \left((k+1)^{p^{rm}} - 1\right)\left((k+1)^{k^m - p^{rm}} + (k+1)^{k^m - 2p^{rm}} + \cdots \right.$$
$$\left. + (k+1)^{p^{rm}} + 1\right),$$

we see that $(k+1)^{k^m} - 1$ is also divisible by $p^{r(m+1)}$.

In particular, since p^r is an arbitrary prime power dividing k, we see that $(k+1)^{k^m} - 1$ is divisible by every prime power that divides k^{m+1}. This is enough to verify that $(k+1)^{k^m} - 1$ is divisible by k^{m+1} itself. ■

Lemma 5. *For each $k \geq 2$, $d_k\alpha_k$ is an integer. In particular, since d_k is a power of k, we see that α_k is a k-adic fraction.*

Proof. We proceed by induction on k, the cases $k = 2$ and $k = 3$ being evident by inspection. For the inductive step, suppose (as our induction hypothesis) that $d_k\alpha_k$ is indeed an integer for a given $k \geq 3$. We may use (5) and (7) to write

$$d_{k+1}\alpha_{k+1} = (d_{k+1} - 1)\alpha_k = \left((k+1)^{d_k/k} - 1\right)\alpha_k = \frac{(k+1)^{d_k/k} - 1}{d_k} d_k\alpha_k. \qquad (11)$$

The factor $d_k\alpha_k$ is an integer by the induction hypothesis. On the other hand, we may rewrite

$$(k+1)^{d_k/k} - 1 = (k+1)^{k^{d_{k-1}/(k-1)-1}} - 1.$$

Applying Lemma 4 with $m = d_{k-1}/(k-1) - 1$, we see that this expression is divisible by $k^{d_{k-1}/(k-1)} = d_k$. Therefore the fraction on the right-hand side of (11) is in fact an integer, and so $d_{k+1}\alpha_{k+1}$ is itself an integer. ■

As mentioned in the introduction, the key to proving that α is irrational is to show that it is in fact a Liouville number. It is a standard fact that any Liouville number is transcendental [**6**, Theorem 7.9]; here is a proof.

Lemma 6. *Every Liouville number is transcendental.*

Proof. We prove the contrapositive: no algebraic number can satisfy the Liouville property (3) for all positive m. Suppose that β is algebraic. Without loss of generality, we may suppose that $|\beta| \leq \frac{1}{2}$ by adding an appropriate integer. Let

$$m_\beta(x) = c_d x^d + c_{d-1}x^{d-1} + \cdots + c_2x^2 + c_1x + c_0$$

be the minimal polynomial for β, where the coefficients c_i are integers. Now suppose that p/q is a rational approximation to β, say $|\beta - p/q| < \frac{1}{2}$. Then

$$
\begin{aligned}
m_\beta\left(\frac{p}{q}\right)
&= m_\beta\left(\frac{p}{q}\right) - m_\beta(\beta) \\
&= c_d\left(\left(\frac{p}{q}\right)^d - \beta^d\right) + \cdots + c_2\left(\left(\frac{p}{q}\right)^2 - \beta^2\right) + c_1\left(\frac{p}{q} - \beta\right) \\
&= \left(\frac{p}{q} - \beta\right)\left(c_d\left(\left(\frac{p}{q}\right)^{d-1} + \left(\frac{p}{q}\right)^{d-2}\beta + \cdots + \beta^{d-1}\right) + \cdots + c_2\left(\frac{p}{q} + \beta\right) + c_1\right).
\end{aligned}
$$

Since neither β nor p/q exceeds 1 in absolute value, we see that

$$\left|m_\beta\left(\frac{p}{q}\right)\right| \le \left|\frac{p}{q} - \beta\right| C(\beta), \tag{12}$$

where we have defined the constant

$$C(\beta) = d|c_d| + (d-1)|c_{d-1}| + \cdots + 2|c_2| + |c_1|.$$

On the other hand, $m_\beta(p/q)$ is a rational number with denominator at most q^d, and it is nonzero since m_β is irreducible. Therefore

$$\left|m_\beta\left(\frac{p}{q}\right)\right| \ge \frac{1}{q^d}. \tag{13}$$

Together, the inequalities (12) and (13) imply that

$$\left|\beta - \frac{p}{q}\right| \ge \frac{C(\beta)^{-1}}{q^d},$$

which precludes the inequality (3) from holding when m is large enough. ∎

To show that α is indeed a Liouville number, we need an inequality somewhat stronger than the one given in Lemma 1. The following lemma furnishes a simple inequality that is strong enough for this purpose.

Lemma 7. $d_{j+1} > d_j^{d_{j-1}}$ *for all* $j \ge 5$.

Proof. It is immediate that

$$
\begin{aligned}
d_{j+1} &= (j+1)^{d_j/j} > j^{d_j/j} > j^{2d_{j-1}^2/j} = \left(j^{d_{j-1}/(j-1)}\right)^{d_{j-1}\cdot 2(j-1)/j} \\
&= d_j^{d_{j-1}\cdot 2(j-1)/j} > d_j^{d_{j-1}},
\end{aligned}
$$

where we have used Lemma 1 for the second inequality. ∎

Lemma 8. *α is a Liouville number; in particular, α is transcendental.*

Proof. We show that the α_k provide the very close rational approximations needed in (3) to make α a Liouville number. Indeed, α_k can be written as a fraction whose denominator is d_k by Lemma 5, while Lemma 2 tells us that for $k \geq 5$

$$0 < |\alpha - \alpha_k| < \frac{2}{d_{k+1}} < \frac{2}{d_k^{d_{k-1}}},$$

where we use Lemma 7 for the last inequality. Since d_{k-1} tends to infinity with k, this shows that α is a Liouville number; Lemma 6 now ensures that it is transcendental. ∎

At last we have all the tools we need to establish the claim advanced in the introduction:

Theorem. *The number α defined in (8) is irrational and absolutely abnormal.*

Proof. Lemma 8 shows that α is irrational. To prove that α is absolutely abnormal, we exploit the fact that for every integer base $b \geq 2$, α is just a tiny bit less than the b-adic fraction α_b. Since the b-ary expansion of α_b terminates in an infinite string of zeros, the slightly smaller number α has a long string of digits equal to $b-1$ before resuming a more random behavior. (This is evident in the decimal expansion (9), since α_5 is a 10-adic fraction as well as a 5-adic fraction.) This happens more than once, as each of $\alpha_b, \alpha_{b^2}, \alpha_{b^3}, \ldots$ is a b-adic fraction. Consequently, the b-ary expansion of α has increasingly long strings consisting solely of the digit $b-1$, which prevents it from being even simply normal to the base b.

More quantitatively, let $b \geq 2$ and r be positive integers. Since $d_{b^r}\alpha_{b^r}$ is an integer by Lemma 5, and since $d_{b^r} = (b^r)^{d_{b^r-1}/(b^r-1)}$ by definition, the b-ary expansion of α_{b^r} terminates after at most $rd_{b^r-1}/(b^r-1)$ nonzero digits. On the other hand, Lemma 2 ensures that α is less than α_{b^r} but by no more than $2/d_{b^r+1} = 2/(b^r+1)^{d_{b^r}/b^r} < 2/b^{rd_{b^r}/b^r}$. Therefore, when we subtract this small difference from α_{b^r}, the resulting b-ary expansion has occurrences of the digit $b-1$ beginning at the $(rd_{b^r-1}/(b^r-1)+1)$-th digit at the latest, and continuing through at least the (rd_{b^r}/b^r-1)-th digit since the difference starts to show only in the (rd_{b^r}/b^r)-th digit at the soonest. Using the notation in (1), this implies that

$$N\left(\alpha; b, b-1, \frac{rd_{b^r}}{b^r}\right) \geq \frac{rd_{b^r}}{b^r} - \frac{rd_{b^r-1}}{b^r-1} - 1 > \frac{rd_{b^r}}{b^r} - \frac{2rd_{b^r-1}}{b^r}.$$

At this point we can calculate

$$\begin{aligned}\limsup_{x\to\infty} x^{-1}N(\alpha; b, b-1, x) &\geq \limsup_{r\to\infty}\left(\frac{b^r}{rd_{b^r}}N\left(\alpha; b, b-1, \frac{rd_{b^r}}{b^r}\right)\right)\\ &\geq \limsup_{r\to\infty}\left(1 - \frac{2d_{b^r-1}}{d_{b^r}}\right).\end{aligned}$$

Using Lemma 1, we see that

$$\limsup_{x\to\infty} x^{-1}N(\alpha; b, b-1, x) \geq \limsup_{r\to\infty}\left(1 - \frac{2d_{b^r-1}}{2d_{b^r-1}^2}\right) = \limsup_{r\to\infty}\left(1 - \frac{1}{d_{b^r-1}}\right) = 1.$$

In particular, the frequency $\delta(\alpha; b, b-1)$ defined in (2) either does not exist or else equals 1, either of which precludes α from being simply normal to the base b. Since $b \geq 2$ was arbitrary, this shows that α is absolutely abnormal. ∎

3. GENERALIZATIONS AND FURTHER QUESTIONS. We mentioned in the introduction that our construction of an irrational, absolutely abnormal number can be generalized to exhibit an uncountable set of absolutely abnormal numbers in any open interval, and we now describe that extension. We may limit our attention to subintervals of [0, 1], since the set of normal numbers to any base is invariant under translation by an integer. In the original construction at the beginning of Section 2, we began with $d_2 = 2^2$ and $\alpha_2 = \frac{3}{4}$; to be more general, let α_2 be any 2-adic fraction a/d_2, where $d_2 = 2^{n_2}$ for some positive integer n_2. Next fix any sequence $n_3, n_4, \ldots$ of positive integers and modify the recursive definition (5) of the d_j to

$$d_j = j^{n_j d_{j-1}/(j-1)} \quad (j \geq 3). \tag{14}$$

If we now set

$$\alpha_k = \alpha_2 \prod_{j=3}^{k} \left(1 - \frac{1}{d_j}\right)$$

(where of course the numbers α_k now depend on α_2 and the n_j), then the presence of the integers n_j in (14) does not invalidate the proof of Lemma 5 that $d_k\alpha_k$ is always a k-adic fraction. Therefore the new limit $\alpha = \lim_{k\to\infty} \alpha_k$ can be shown to be a transcendental, absolutely abnormal number in exactly the same way, the modifications only accelerating the convergence of the infinite product and enhancing the ease with which the various inequalities in the Lemmas are satisfied. (When we make this modification, the one case we must avoid is $\alpha_2 = \frac{1}{2}$ and $n_2 = n_3 = \cdots = 1$, for in this case it happens that $d_j = j^1$ for every $j \geq 2$. Then each product α_k is a telescoping product with value $1/k$, and their limit $\alpha = 0$, while certainly absolutely abnormal, is uninterestingly so.)

In particular, Lemma 2 applied with $k = 2$ would show in this context that $a/2^{n_2} > \alpha > (a-2)/2^{n_2}$; thus by choosing $\alpha_2 = a/2^{n_2}$ appropriately, we can ensure that the resulting number α lies in any prescribed open subinterval of [0, 1]. Moreover, the various choices of the integers $n_3, n_4, \ldots$ give rise to distinct limits α; one can show this by considering the first index $j \geq 3$ at which the choices of n_j differ, say, and then applying Lemma 2 with $k = j$ to each resulting α. This generalization thus permits us to construct uncountably many transcendental, absolutely abnormal numbers in any prescribed open interval.

One interesting special case of this generalized construction arises from the choices $\alpha_2 = \frac{1}{2}$ and $n_j = \phi(j-1)$ for all $j \geq 3$, where ϕ is the Euler totient function; these choices give the simple recursive rule $d_j = j^{\phi(d_{j-1})}$ for $j \geq 3$. In this special case, the crucial property that $d_k\alpha_k$ is always an integer is in fact a direct consequence of Euler's theorem that $a^{\phi(n)}$ is always congruent to 1 modulo n as long as a and n have no common factors. In general, the smallest exponent e_k we can take in the recursive rule $d_k = k^{e_k}$ so that this crucial property is satisfied is the multiplicative order of k modulo d_{k-1}, which might be smaller than $d_{k-1}/(k-1)$; however, our construction given in Section 2 has the advantage of being more explicit, as it is not necessary to wait and see the exact value of d_{k-1} before knowing how to construct d_k.

Schmidt [7] actually proved a stronger result than mentioned earlier. Given any set S of integers exceeding 1 with the property that an integer b is in S if and only if every perfect power of b is in S, Schmidt established the existence of real numbers that are normal to every base $b \in S$ and abnormal to every base $b \notin S$. (Our problem is the special case where $S = \emptyset$.) It would be interesting to see if our construction could be modified to produce these "selectively normal numbers" as well.

We conclude with a few remarks about *absolutely simply abnormal numbers*, numbers that are simply normal to no base whatsoever. As we saw in the proof of our Theorem, the number α does meet this stronger criterion of abnormality. On the other hand, while all rational numbers are absolutely abnormal, many of them are in fact simply normal to various bases. For example, $1/3$ is simply normal to the base 2, as its binary representation is $0.010101\ldots$. In fact, one can check that every fraction in reduced form whose denominator is $3, 5, 6, 9, 10, 11, 12, 13, 17, 18, 19, 20, \ldots$ is simply normal to the base 2 (presumably there are infinitely many such odd denominators—can it be proved?); every fraction whose denominator is $7, 14, 19, 21, 31, \ldots$ is simply normal to the base 3; and so on. Somewhat generally, if p is a prime such that one of the divisors b of $p-1$ is a primitive root for p, then every fraction whose denominator is p is simply normal to the base b (although this is not a necessary condition, as the normality of fractions with denominator 17 to base 2 shows).

For a fraction with denominator q to be simply normal to the base b (we can assume, by multiplying by b a few times if necessary, that b and q are relatively prime), it is necessary for b to divide the multiplicative order of b modulo q, and hence b must certainly divide $\phi(q)$ by Euler's theorem. Therefore, every fraction whose denominator is a power of 2 is absolutely simply abnormal. One can also verify by this criterion that every fraction whose denominator in reduced form is 15 or 28, for example, is absolutely simply abnormal. It seems to be a nontrivial problem to classify, in general, which rational numbers are absolutely simply abnormal. Since some fractions with denominator 63 are simply normal to the base 2 while others are not, absolute simple abnormality probably depends in general on the numerator as well as the denominator of the fraction.

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