WHAT IS A CONVEX SET?

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This is a slight expansion (and, of course, a translation) of the author's article [63] on *Convexité* in the new French encyclopedia, *Encyclopedia Universalis*, and appears here with the kind permission of the encyclopedia's publishers. Its purpose is to supply a broad but brief survey, at a rather elementary level, of several aspects of convexity theory. Those aspects are emphasized which appear to the author to be most active at present and to be most accessible to the chosen level of exposition. In the allotted space, the topics covered cannot, of course, be "surveyed" in the usual sense; instead, each is represented by one or more of its highlights. No proofs are included.

For theorems which are often designated in the literature by authors' names, the relevant names are given here, even though in some cases the implied attribution is incomplete. Beyond that, there has been no attempt at attribution. The references are in most cases not to the original or the most definitive sources, but rather to expository treatments or to papers which contain useful collections of additional references. While this policy probably maximizes the utility/size ratio of the bibliography, it has the unfortunate consequence of omitting the names of many prominent workers in the field. Those names can be found in the bibliographies of the references cited. A reference which appears in parentheses at the end of a paragraph contains useful information, or at least useful references, for the entire area of convexity theory with which the paragraph is concerned. In addition to items which are specifically mentioned in the text, the bibliography includes a number of books, monographs, lecture notes, symposium volumes, and survey articles in which the notion of convexity has played an important role. Most of these have appeared in the past fifteen years.

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Introduction. The study of **convex sets** is a branch of geometry, analysis, and linear algebra that has numerous connections with other areas of mathematics and serves to unify many apparently diverse mathematical phenomena. It is also relevant to several areas of science and technology.

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Though convex sets are defined in various settings (see [27] for a survey), the most useful definitions are based on a notion of betweenness. When E is a space in which such a notion is defined, a subset C of E is called **convex** provided that for each two points x and y of C, C includes all points between x and y. The most important setting, and the only one to be discussed here, is that in which E is a vector space over the real number field R or, in particular, is the n-dimensional Euclidean space E^n, and the points between x and y are those of the line segment xy. Thus, a subset C of a real vector space is convex provided that C contains every segment whose endpoints both belong to C. (For example, a cube in E^3 is convex but its boundary is not, for the boundary does not contain the segment xy unless x and y lie together in some 2-dimensional face of the cube.) The importance of convexity theory stems from the fact that convex sets arise frequently in many areas of mathematics and are often amenable to rather elementary reasoning. Even the infinite-dimensional theory is based to a considerable extent on 2- and 3-dimensional reasoning.

The first systematic study of convexity was made by Minkowski (1864–1909), whose works [71] contain, at least in germinal form, most of the important ideas of the subject. The early developments of convexity theory were finite-dimensional and directed mainly toward the solution of quantitative problems; an excellent survey of them was made by Bonnesen and Fenchel [14] in 1934. Since 1940, however, the combinatorial, qualitative, and dimension-free parts of the theory have tended to predominate, perhaps because of their many applications in other areas of mathematics. After some preliminary material that is relevant to all parts of the theory, the present exposition begins with the quantitative and combinatorial aspects because they are restricted to the finite-dimensional spaces that are most likely to be familiar to the reader. In discussing, later, the qualitative and dimension-free aspects of the theory, some slight familiarity with topological vector spaces is assumed. The reader who lacks this familiarity may restrict his attention to the case of Euclidean n-space E^n.

A fascinating aspect of convexity theory is the large number of easily stated and intuitively appealing unsolved problems that it still contains. A few such problems are included here.

Preliminary Material. Any two distinct points x and y of a real vector space E determine a unique **line.** It consists of all points of the form $(1-\lambda)x+\lambda y$, λ ranging over all real numbers. Those points for which $\lambda \geqq 0$ and for which $0 \leqq \lambda \leqq 1$ form respectively the **ray** from x through y and the **segment** xy. An **affine** set is one that contains all lines determined by pairs of its points; equivalently, it is a translate of a linear subspace. For example, the affine sets in E^3 are the empty set, the onepointed sets, the lines, the planes, and E^3 itself. A **hyperplane** H in E is an affine set of deficiency or codimension 1; that is, H is not properly contained in any affine subset of E other than E itself. In particular, the hyperplanes of E^n are its affine subsets of dimension $n-1$. For any hyperplane H, the complement $E \sim H$ is expressible in a unique way as the union

of two convex sets. They are called the **open halfspaces** bounded by H and their unions with H are the **closed halfspaces** bounded by H. Two sets X and Y are said to be **separated** by H provided that X lies in one of these closed halfspaces and Y in the other. The set X is **supported** by H at the point x provided that x belongs to X but is separated from X by H. Fig. 1 shows a hyperplane H (in this case, a line) in E^2, separating the convex sets X and Y and supporting X at the point x.

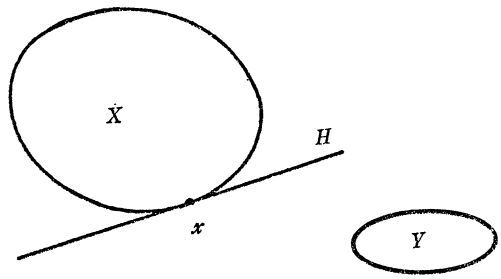

FIG. 1

Intimately related to the notion of a convex set is that of a convex function, which is important in most parts of convexity theory and in several areas of analysis. Let ϕ be a real-valued function whose domain D lies in a real vector space E. Then ϕ is called **convex** provided that D is convex and ϕ satisfies the inequality,

$$\phi((1-\lambda)x+\lambda y) \leqq (1-\lambda)\phi(x)+\lambda\phi(y),$$

for all points x and y of D and all numbers λ between 0 and 1. Equivalently, ϕ is a convex function if and only if its **epigraph** G is a convex set, where G is the subset of the product space $E\times R$ consisting of all ordered pairs (x, τ) such

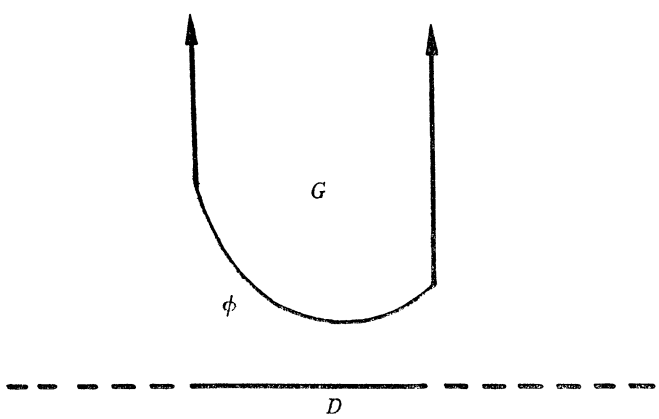

FIG. 2

that $x \in D$ and $\tau \geqq \phi(x)$. Fig. 2 shows a convex function ϕ, its domain D, and its epigraph G.

A function is called **concave** provided that its negative is convex, and **affine** provided that it is both convex and concave. A real-valued function on E is affine if and only if it differs by a constant from a linear function. The hyperplanes H in E are precisely the zero sets of the nonconstant affine functionals f on E. If H is the set of all x for which $f(x) = 0$, then the closed halfspaces bounded by H are determined by the inequalities $f(x) \leqq 0$ and $f(x) \geqq 0$.

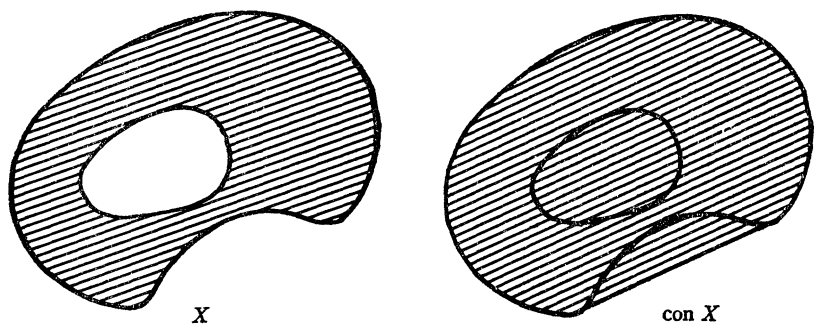

FIG. 3

The **convex hull** of a set X, denoted here by con X, is the intersection of all convex sets containing X. It is convex, as is any intersection of convex sets, and hence is the *smallest* convex set containing X. (Fig. 3 shows a nonconvex plane set and its convex hull.) Equivalently, con X is the set of all **convex combinations** of X—that is, points of the form $\sum_1^k \lambda_i x_i$, where the x_i's are points of X and the λ_i's are positive numbers whose sum is 1. For any such combination and for any convex function ϕ whose domain contains X,

$$\phi\left(\sum_1^k \lambda_i x_i\right) \leqq \sum_1^k \lambda_i \phi(x_i).$$

The **closed convex hull** of a set is the closure of its convex hull.

A point x of a convex set C is called an **extreme point** of C provided that $C \sim \{x\}$ is convex or, equivalently, that x is not an inner point of any segment in C. More generally, a **face** of C is a convex set $F \subset C$ such that F is not "crossed" by any segment in C—that is, $xy \subset F$ whenever x and y belong to C and F includes an inner point of xy. (For example, a cube in E^3 has six 2-faces, twelve 1-faces (edges), and eight 0-faces (extreme points). It is the convex hull of its set of extreme points.)

The terms *body* and *cone* are used in different ways by various authors. Here **body** always means a bounded closed subset of E^n that has nonempty interior, and **cone** means a set in a real vector space which is a union of rays from the origin O.

Quantitative Aspects. Convexity is a basic notion in the so-called geometry of numbers, and, indeed, it was the latter subject that led Minkowski to many of his investigations. One of his most striking results is that if C is a convex body in E^n which is symmetric about the origin O (that is, $x \in C$ implies $-x \in C$), and if the volume $V(C)$ is at least 2^n, then C includes at least one point other than O whose coordinates are all integers [21].

A **packing** of convex bodies is an arrangement in which no two of the bodies have common interior points. In addition to being of interest for themselves, packing problems are found in number theory, information theory, crystallography, botany, virology, and other areas of science. Often the interest is in packings of maximum density. The densest packing of congruent circular disks in E^2 is one in which the disks are inscribed in nonoverlapping regular hexagons covering E^2; each disk touches six others. (See Fig. 4.) It has long been conjectured that a densest packing of congruent spherical balls in E^3 is the cubic close-packing of Kepler, obtained by imagining the space to be divided into black and white cubes forming a 3-dimensional chessboard, and then placing a ball concentric with each black cube and tangent to each of the twelve edges of the cube; each ball touches twelve others. However, the conjecture has been proved only for packings of congruent balls whose centers form a lattice (if x and y are centers then so is $2x-y$). ([35] [81])

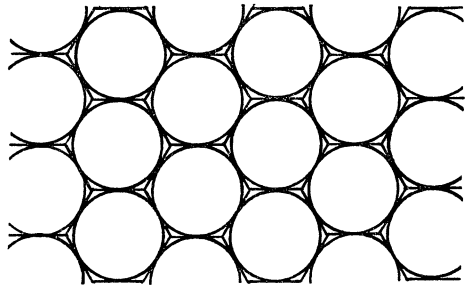

FIG. 4

There is a rich collection of quantitative results involving such measurements of convex bodies as volume, surface area, diameter, etc. Many extremal problems concerning such measurements have spherical balls or simplices as their solutions. (A **simplex** in E^n is the convex hull of $n+1$ points not contained in any hyperplane.) For example, the isoperimetric inequality asserts that if S and V are respectively the surface area and the volume of a convex body C in E^n, and if ω is the volume of an n-dimensional ball of unit radius, then $S^n \geqq n\omega V^{n-1}$, with equality if and only if C is a ball. Thus of all bodies with given volume, balls have the least surface area [13] [14] [46] [47]. Any convex body C of E^n lies in a unique ball of minimum radius r. Jung's inequality, of interest in approximation theory, asserts that if C's diameter is d then $r \leqq (n/(2n+2))^{1/2}d$, with equality if and only if C is a regular simplex [27] [47]. Loewner's theorem

asserts that any body in E^n lies in a unique ellipsoid of minimum volume [**28**].

Convex sets are prominent in the theory of geometric probability. For example, the following problem was posed (in a different form) by Sylvester. Let C be a convex body of unit volume in E^n and let $n+1$ points be chosen from C, independently and at random. Except in degenerate cases, the convex hull of these points is an n-simplex. What is the expected volume, V_C, of the simplex? For $n=2$, the values of V_C are between $1/12$ and $35/48\pi^2$, attained respectively when C is a triangle and when C is an ellipse. For larger values of n, V_C is known when C is a ball but not when C is a simplex [**61**].([**55**] [**72**][**73**])

For convex bodies $C_1, \cdots, C_k$ in E^n and positive numbers $\lambda_1, \cdots, \lambda_k$, the set of all points of the form $\lambda_1 x_1 + \cdots + \lambda_k x_k$ with $x_i \in C_i$ is another convex body C, denoted by $\lambda_1 C_1 + \cdots + \lambda_k C_k$. When $C_1, \cdots, C_k$ are fixed, the volume of C is expressible as a homogeneous polynomial of degree n in the parameters $\lambda_1, \cdots, \lambda_k$. Some of the deepest parts of the quantitative theory concern the coefficients of this polynomial, which are called the **mixed volumes** of $C_1, \cdots, C_k$. A basic tool in the study of mixed volumes is the Brunn-Minkowski theorem asserting that for $0<\lambda<1$, $(V((1-\lambda)C_1+\lambda C_2))^{1/n} \geqq (1-\lambda)(V(C_1))^{1/n}+\lambda(V(C_2))^{1/n}$ (that is, the nth root of the volume is a concave function of λ) and characterizing the cases of equality. Inequalities for mixed volumes yield the isoperimetric inequality and other inequalities of immediate geometric interest. ([**14**] [**46**] [**47**])

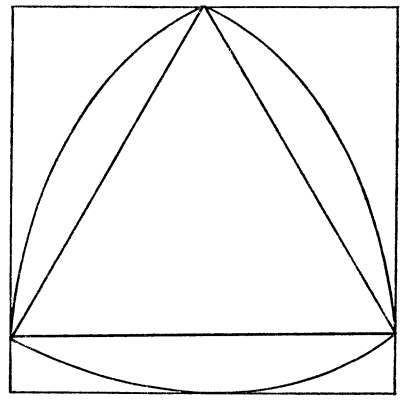

FIG. 5

A convex body C in E^n is said to be of **constant breadth** b provided that b is the distance between any two parallel supporting hyperplanes of C; equivalently, C is of diameter b and the diameter is increased by adding any point of E^n not in C. From the second description it follows that any set of diameter $\leqq b$ in E^n lies in at least one convex body of constant breadth b. Hence the following problem of Borsuk can be reduced to the case in which X is a convex body of constant breadth: Can every set X of diameter 1 in E^n be covered by

$n+1$ sets of diameter <1? The answer is affirmative for $n \leqq 3$, unknown for $n>3$ [42]. Noncircular plane convex bodies of constant breadth have been studied by many mathematicians. Their special properties have led to their use in kinematic linkages and in other mechanisms. Any such body can be placed in a square and then "rotated" while remaining in contact with all four sides of the square. (See Fig. 5.) ([5] [14] [87])

Having mentioned some unsolved problems in E^3 and E^4, we end this section with one in E^2. A **chord** of a convex body is a segment joining two boundary points, and an **equichordal point** is one through which all chords are of equal length. Does any plane convex body have two equichordal points? [60]

Combinatorial Aspects. Much of combinatorial convexity theory deals with intersection properties of convex sets. The intersection C of any family of convex sets is itself convex, though C may be empty. Helly's theorem asserts that C is nonempty if the convex sets are all in E^n, each $n+1$ of them have nonempty intersection, and the family is finite or its members are all compact. There are numerous generalizations and applications of Helly's theorem. From its 1-dimensional form it follows that if C is a cube in E^n then any family of pairwise intersecting translates of C has nonempty intersection. In fact, a convex body C has this intersection property if and only if C is affinely equivalent to a cube. ([27] [48] [87])

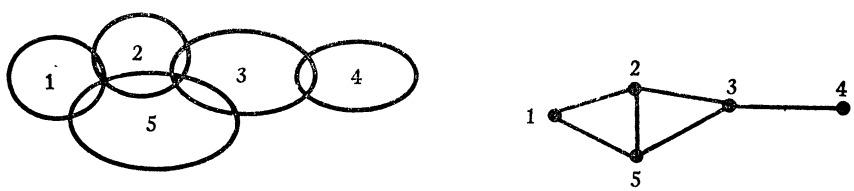

FIG. 6

The problem of determining all intersection properties of convex sets is not trivial even in E^1. It leads to the notion of an interval graph, which has been used in such diverse fields as molecular genetics, psychophysics, archaeology, and ecology. For any family of sets the associated **intersection graph** is an abstract graph having a node for each member of the family, two nodes being joined by an arc of the graph if and only if the corresponding sets intersect. Fig. 6 shows a family of convex sets and its intersection graph. Any finite graph can be realized as the intersection graph of a family of convex bodies in E^3, but not so in E^2 or E^1. An **interval graph** is one that is the intersection graph of a finite family of convex bodies in E^1. Such graphs have been characterized in various ways, but the corresponding problem relative to E^2 is still open. ([84] [86])

Another area of combinatorial research is concerned with the representation of convex hulls. The simplest and most useful result is Carathéodory's theorem, asserting that if $X \subset E^n$ and $u \in \text{con } X$ then $u \in \text{con } Y$ for some set Y consisting

of $n+1$ or fewer points of X. For example, when $X \subset E^2$ any point of con X belongs to X, to a segment determined by two points of X, or to a triangle determined by three points of X. Carathéodory's theorem has many generalizations and applications, including the fact that con X is compact for each compact $X \subset E^n$. ([**27**] [**79**])

The most extensive combinatorial developments deal with the facial structure of convex polyhedra. Though terminology has not been standardized, we here use the term **polyhedron** to mean a subset of E^n that is the intersection of a finite number of closed halfspaces. Of special interest are the bounded polyhedra, here called **polytopes,** and the polyhedral cones. By the lemma of Farkas, a set is a polytope if and only if it is the convex hull of a finite set of points, and is a polyhedral cone if and only if it is the convex hull of a finite number of rays from the origin. More generally, the following five conditions on a set P in E^n are equivalent: (i) P is a polyhedron; (ii) P is a closed convex set whose number of faces is finite; (iii) P is the convex hull of a finite system of points and rays; (iv) P is the vector sum $B+C=\{b+c\colon b \in B,\ c \in C\}$ of a polytope B and a polyhedral cone C; (v) P is the closed convex hull of the union of a polytope B and a translate of a polyhedral cone C. Fig. 7 shows a 2-dimensional polyhedron P and the associated sets B and C. ([**56**])

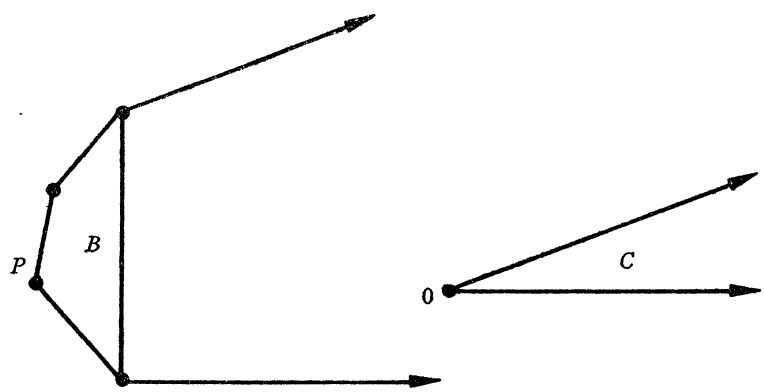

FIG. 7

In the combinatorial study of polyhedra, the first landmark was Euler's 1752 theorem asserting that $v-e+f=2$ for any 3-polytope, where v, e, and f are respectively the numbers of vertices, edges, and 2-faces. (Extreme points of polyhedra are usually called **vertices.**) The generalization of Schläfli and Poincaré asserts that if $f_i(P)$ is the number of i-dimensional faces of an n-dimensional polytope P, then $\sum_{i=0}^{n-1} (-1)^i f_i(P) = 1-(-1)^n$. The second landmark was Steinitz's 1934 theorem characterizing the graphs of 3-polytopes, the combinatorial structures formed by vertices and edges, as those that are planar (representable in E^2 without crossings) and 3-connected (between any two vertices

there are three independent paths). The first graph of Fig. 8 corresponds to a cube. The second and third graphs of Fig. 8 do not correspond to any 3-polytope, for the first is not 3-connected and the second is not planar. However, the third graph does correspond to a 4-dimensional simplex. Various properties, including n-connectedness, have been established for the graphs of n-polytopes, but no combinatorial characterization is known for $n>3$. ([**43**])

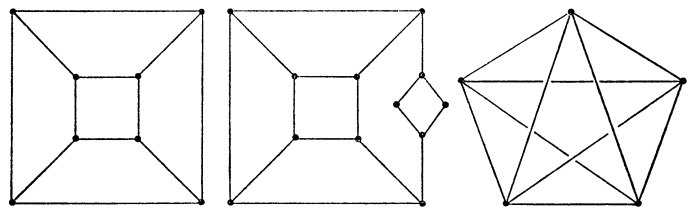

FIG. 8

A third landmark in the combinatorial study of polyhedra was the development, beginning in the late 1940's and still continuing, of computational techniques for minimizing linear functions on polyhedra. These techniques, known as *linear programming*, provide solutions to a wide range of practical optimization problems, and they are also useful for other computations involving polyhedra. Their importance led to a renewed interest in polyhedra and, for example, to rediscovery of the striking fact that for each $k>n$ there is an n-polytope with k vertices such that each $[n/2]$ vertices determine a face. A closely related development is the recent proof [**70**] that the maximum number of vertices possessed by any n-polyhedron with k $(n-1)$-faces is

$$\binom{k-[(n+1)/2]}{k-n}+\binom{k-[(n+2)/2]}{k-n}. \qquad ([26]\ [43]\ [45]\ [59])$$

Qualitative Aspects. The topics to be discussed in this section include normed vector spaces, polarity, separation and support theorems, extreme point theorems, and fixed point theorems.

For a real-valued function ϕ on a real vector space E, any two of the following conditions imply the third: subadditivity ($\phi(x+y)\leqq\phi(x)+\phi(y)$ for all x, $y\in E$); positive homogeneity ($\phi(\lambda x)=\lambda\phi(x)$ for all $x\in E$ and $\lambda\geqq 0$); convexity. A **norm** is a function that satisfies these conditions as well as being symmetric ($\phi(-x)=\phi(x)$) and positive ($x\neq 0$ implies $\phi(x)>0$). A **normed vector space** consists of a vector space E together with a norm on E. The norm is usually denoted by $\|\ \|$ and leads to a useful notion of distance by defining the distance between x and y as $\|x-y\|$. The **unit ball** of such a space is the set of all points x for which $\|x\|\leqq 1$, while the **unit sphere** is defined by the condition $\|x\|=1$. When $r=2$, the function $\|x\|_r=(\sum_1^n |x_i|^r)^{1/r}$ is the usual Euclidean norm for E^n; an inequality of Minkowski asserts that it is a norm for all $r\geqq 1$. In general, the notion of convexity plays a key role in the theory of inequalities, and many useful

inequalities assert merely that a certain function is convex. Another important norm for E^n is given by

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i| .$$

The unit ball for $\| \ \|_\infty$ is an n-dimensional cube and for $\| \ \|_1$ is a so-called cross-polytope (a regular octahedron when $n=3$). Fig. 9 shows the unit spheres in E^2 associated with $\| \ \|_r$ for $r=1$, $3/2$, 2, 3, and ∞. ([29])

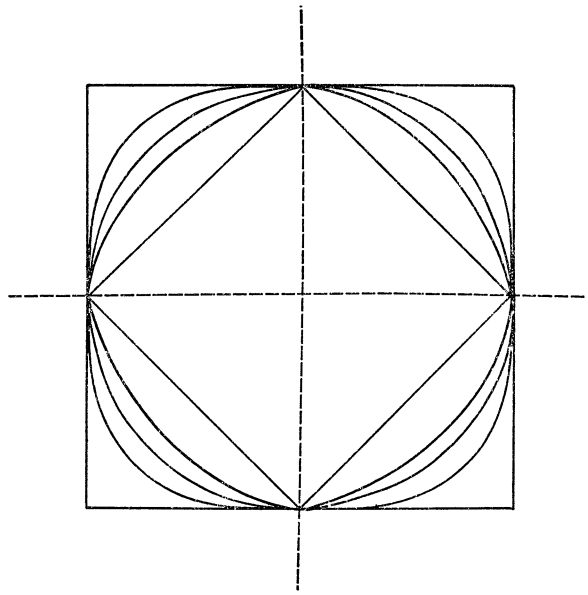

FIG. 9

The study of normed vector spaces is in a sense equivalent to the study of a certain class of convex sets. The equivalence involves the notion of the **gauge function** μ of a set U, where, for each $x \in E$, $\mu(x)$ is defined as the infimum of all numbers $\lambda > 0$ such that $x \in \lambda U$. For any norm $\| \ \|$ on E, the associated unit ball U is a convex set that intersects every line through O in a closed segment having O as its midpoint; further, $\| \ \|$ is the gauge function of U. Conversely, for any convex set U of the sort described, the associated gauge function is a norm for which U is the unit ball. Thus any property of a normed vector space can be described completely in terms of its unit ball or its unit sphere. For example, the **rotundity** of a normed space may be defined by saying that $\|x+y\| < \|x\| + \|y\|$ whenever x and y are not collinear with O, or by saying, equivalently, that the unit sphere does not contain any line segments. **Smoothness** of the space E may be defined by saying that for any two points x and y not equal to 0, the function $\phi(\lambda) = \|x+\lambda y\|$ is differentiable at $\lambda = 0$—or by saying, equivalently, that the unit sphere has at each point a unique supporting hyperplane. Euclidean n-space is both rotund and smooth. The behavior of a normed space can sometimes be

improved by **renorming,** which means introducing a new norm that is caught between two positive multiples of the original norm and hence induces the same topology on the space. For example, any separable normed linear space can be renormed so as to be simultaneously smooth and strictly convex. ([24] [25] [29])

All of the quantitative problems mentioned earlier for Euclidean spaces have been studied also for finite-dimensional normed spaces, commonly called **Minkowski spaces.** The analogue of Jung's inequality asserts that if E is an n-dimensional Minkowski space and X is a set of diameter d in E, then there is a point z of E such that $\|x-z\| \leqq (n/(n+1))d$ for all $x \in X$ [27]. In contrast to the Euclidean case, the unit ball of a Minkowski space need not be an extreme body so far as the isoperimetric problem for that space is concerned [18]. There are many characterizations of those Minkowski spaces which are equivalent to Euclidean spaces or, in other words, whose unit balls are ellipsoids. Most of these characterizations are related to the fact that, among the Minkowski spaces E of dimension $\geqq 3$, each of the following conditions characterizes the Euclidean spaces: (a) (Jordan—vonNeumann) $\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2$; (b) (Blaschke-Kakutani) for any hyperplane H through O in E, there is a linear projection of norm 1 of E onto H–equivalently, there is a line L such that if U is the unit ball then the intersection $U \cap H$ is equal to the intersection of U with the "cylinder" $U+L$. ([29] [58])

So far as the applications of convexity in other parts of mathematics are concerned, separation and support theorems are of special importance. They are widely used in functional analysis and have been used in game theory, in the theory of summability, and even to prove certain coloring theorems of graph theory. Together with Lyapunov's theorem asserting the convexity of the range of a nonatomic vector-valued measure [9] [49] [68], they are among the principal abstract tools of the theory of optimal control [62]. Separation theorems set forth conditions under which two nonempty disjoint convex subsets X and Y of a topological vector space E can be separated by a hyperplane, either in the weak sense defined above or in various stronger senses. It suffices, for example, that E should be finite-dimensional or that one of the sets should have nonempty interior. A consequence is that if C is a convex set whose interior is empty and A is a nonempty affine set disjoint from the interior of C, then A lies in a hyperplane separating A from C. In particular, a closed convex set C with nonempty interior is supported at each of its boundary points (Mazur-Bourgin). Though that conclusion may fail if C's interior is empty, the support points of C are dense in C's boundary if E is a Banach space (Bishop-Phelps) and also if E is locally convex and C weakly compact. If A is an affine subset of a real vector space E, ϕ is a convex function on E, and f is an affine function on A such that $f \leqq \phi$ on A, then f can be extended to an affine function g on E with $g \leqq \phi$ on E. This result, a slight improvement of the classical Hahn-Banach theorem of functional analysis, follows from the separation theorem applied to the graph of f and the epigraph of ϕ. Because of this and other relationships, separation and support

theorems may be regarded as geometric relatives of the Hahn-Banach theorem. ([16] [29] [54] [62] [64] [80] [83])

The notion of polarity is essential in convexity theory and in the theory of topological vector spaces. Let E and F be two spaces that are paired by a bilinear form $\langle\ ,\ \rangle$. For example, let $E=F=E^n$ and let $\langle\ ,\ \rangle$ be the usual inner product given by

$$\langle (x_1, \cdots, x_n), (y_1, \cdots, y_n)\rangle = x_1y_1 + \cdots + x_ny_n.$$

For any $X\subset E$ the **polar** X^0 of X is the set of all $y\in F$ such that $\langle x, y\rangle \leqq 1$ for all $x\in X$. The polar is always convex, being an intersection of halfspaces. In geometry this notion plays two roles that are dual to each other. When unable to prove directly a theorem about sets $X_1, X_2, \cdots$, one may find it possible to prove an equivalent statement about $X_1^0, X_2^0, \cdots$. On the other hand, having proved an interesting theorem about $X_1, X_2, \cdots$, one may find that the polar form of the theorem is also of interest. For $1\leqq r$ and $1/r+1/s=1$, the $\|\ \|_r$ unit ball of E^n is polar to the $\|\ \|_s$ unit ball of E^n. Some polar pairs of this sort appeared in Fig. 9. There are close relationships among the notion of the polar of a convex set, the so-called support function of a convex body, and the notion of the conjugate of a normed linear space. ([16] [29] [43] [54] [64])

Both convex and concave functions occur often in practical optimization problems, and both have properties that are helpful in such problems. Let f be a continuous real-valued function whose domain D lies in a locally convex topological vector space E. If f is convex, then any local minimum x_0 for f is a global minimum [80] [83] [88]. That is to say, if there is a neighborhood U of x_0 such that $f(x_0)\leqq f(x)$ for all $x\in U\cap D$, then the same inequality holds for all $x\in D$. This justifies various iterative procedures for finding or approximating x_0. [88] If f is concave and D compact, then f attains a minimum at an extreme point of D. That is one of the reasons for the importance of extreme points in functional analysis. The other reason is contained in theorems of Krein and Milman, asserting that if C is a compact convex subset of a locally convex space, and if $X\subset C$, then C is equal to the closed convex hull of X if and only if the closure of X includes all extreme points of C. Thus C's extreme points form the smallest set by means of which, using convex combinations, all points of C can be approximated. There are extensions of the Krein-Milman theorem to certain noncompact sets, and some sharpening is possible in the finite-dimensional case. For example, if C is a closed convex set in E^n and C contains no line, then C is the convex hull of its extreme points together with its extreme rays. (An **extreme ray** of C is a ray not "crossed" by any segment.) ([3] [6] [22] [29] [54] [64] [76])

When C is a compact convex set in a locally convex space E and X is the set of all extreme points of C, it follows from the Krein-Milman theorem that each point p of C is the barycenter of a probability measure μ carried by the closure $\overline{X}$ of X—that is,

$$f(p) = \int_{x\in\overline{X}} f(x)d\mu(x)$$

for all continuous linear functionals f on E. Several of the integral representation theorems of analysis are consequences of this. However, when X is not closed, it is desirable to have the sharper representation afforded by a measure that is carried by X rather than $\overline{X}$. Choquet's theorem, which has stimulated much research in recent years, asserts that such a representation is always possible when C is metrizable. Uniqueness of the representation, for all $p \in C$, is associated with a useful dimension-free notion of simplex. These ideas have been applied in several fields—for example, in potential theory and in the theory of operator algebras. ([3] [22] [41] [76])

In conclusion, we turn briefly to fixed-point theorems for convex sets, stating two of the simplest but most important ones. Both have been extended in many ways. The Brouwer-Schauder-Tychonov theorem asserts that if C is a compact convex set in a locally convex space and if ϕ is a continuous mapping of C into C, then there is at least one point p of C such that $\phi(p)=p$. The theorem and its relatives are used in many ways, such as proving existence theorems for differential and integral equations, minimax theorems for game theory, and various geometric properties of convex sets. (For example, a compact subset of E^n or of Hilbert space H is convex if and only if each point of the space admits a unique nearest point in the set. It is unknown whether "compact" may be replaced by "closed" in H, though it may be in E^n.) A recent computational development [66] makes it possible to regard this fixed-point theorem as a tool for *constructing* solutions of various sorts of systems, rather than merely establishing their existence. ([15] [30])

The Markov-Kakutani theorem asserts that if C is a compact convex subset of a topological vector space and if Φ is a commuting family of continuous affine transformations of C into C, then there is at least one point $p \in C$ such that $\phi(p)=p$ for all $\phi \in \Phi$. It is used to prove the existence of invariant means on commutative groups, of a finitely additive translation-invariant extension of Lebesgue measure to all bounded subsets of E^n, and in many other ways. Note that local convexity is not required for the Markov-Kakutani theorem. It is unknown whether the assumption can be abandoned in the case of the Krein-Milman extreme point theorem and the Tychonov fixed-point theorem. ([15] [16] [30])

References

1. A. D. Alexandrov, Die innere Geometrie der konvexen Flächen, Akademie-Verlag, Berlin, 1955. (Translated from the 1948 Russian edition.)

2. ———, Konvexe Polyeder, Akademie-Verlag, Berlin, 1958. (Translated from the 1950 Russian edition by H. Abetz, under the scientific editorship of W. Süss.)

3. E. Alfsen, Convex Compact Sets and Boundary Integrals, Springer, Berlin, 1971.

4. R. V. Benson, Euclidean Geometry and Convexity, McGraw-Hill, New York, 1966.

5. L. Beretta and A. Maxia, Insiemi convessi e orbiformi, Univ. Roma e Ist. Naz. Alta Mat. Rend. Mat. (5) 1 (1940) 1–64.

6. C. Berge, Espaces Topologiques: Fonctions Multivoques, Dunod, Paris, 1959.

7. C. Berge and A. Ghouila-Houri, Programming, Games, and Transportation Networks, Methuen, London and Wiley, New York, 1965. (Translated by M. Merrington and C. Ramanujacharyulu from the 1962 French edition.)

8. W. Blaschke, Kreis und Kugel, Teubner, Leipzig, 1916. (Reprinted in 1949 by Chelsea, New York.)

9. E. Bolker, A class of convex bodies, Trans. Amer. Math. Soc., 145 (1969) 323–345.

10. V. G. Boltyanskiĭ and I. Ts. Gohberg, Theorems and Problems of Combinatorial Geometry, (Russian) Nauka, Moscow, 1965.

11. V. G. Boltyanskiĭ and I. M. Yaglom, Convex figures and bodies (Russian), Encyclopedia of Elementary Mathematics, Nauka, Moscow, 1966, pp. 182–269.

12. V. G. Boltyanskiĭ and I. M. Yaglom, Geometric problems on maxima and minima (Russian), Encyclopedia of Elementary Mathematics, Nauka, Moscow, 1966, pp. 270–348.

13. T. Bonnesen, Les Problèmes des Isopérimétriques et des Isépiphanes, Gauthier-Villars, Paris, 1928.

14. ——— and W. Fenchel, Theorie der konvexen Körper, Springer, Berlin 1934. (Reprinted by Chelsea, New York, 1948.)

15. F. Bonsall, Lectures on Some Fixed Point Theorems of Functional Analysis, Tata Institute of Fundamental Research, Bombay, 1962.

16. N. Bourbaki, Espaces Vectoriels Topologiques (rev. ed.), Hermann, Paris, 1966, Chaps. 1–2 and 1967, Chaps. 3–5.

17. M. Brückner, Vielecke und Vielfläche, Teubner, Leipzig, 1900.

18. H. Busemann, The foundations of Minkowskian geometry, Comment. Math. Helv., 24 (1950) 156–187.

19. ———, Convex Surfaces, Interscience, Wiley, New York, 1958.

20. ———, The Geometry of Geodesics, Academic Press, New York, 1955.

21. J. W. S. Cassels, An Introduction to the Geometry of Numbers, Springer, Berlin, 1959.

22. G. Choquet, Lectures on Analysis (ed. by J. Marsden, T. Lance, and S. Gelbart), Benjamin, New York, 1969.

23. H. S. M. Coxeter, Regular Polytopes, Pitman, London, 1948. (Second ed. Macmillan, New York and Collier-Macmillan, London, 1963.)

24. D. F. Cudia, Rotundity, in Convexity, Proc. Sympos. Pure Math., Amer. Math. Soc., 7 (1963) 73–97.

25. ———, The geometry of Banach spaces. Smoothness, Trans. Amer. Math. Soc., 110 (1964) 284–314.

26. G. B. Dantzig, Linear Programming and Extensions, Princeton University Press, Princeton, 1963.

27. L. Danzer, B. Grünbaum, and V. Klee, Helly's theorem and its relatives, in Convexity, Proc. Sympos. Pure Math., Amer. Math. Soc., 7 (1963) 101–180. (Also, an expanded version of this, translated into Russian by S. Zalgaller under the editorship of I. M. Yaglom, published as Helly's Theorem and its Applications (Russian), *Mir*, Moscow, 1968.)

28. L. Danzer, D. Laugwitz, and H. Lenz, Über das Löwnersche Ellipsoid und sein Analogon unter den einem Eikörper einbeschriebenen Ellipsoid, Arch. Math. (Basel), 8 (1957) 214–219.

29. M. M. Day, Normed Linear Spaces, Springer, Berlin, 1958.

30. N. Dunford and J. T. Schwartz, Linear Operators, Part I: General Theory, Interscience, New York, 1958.

31. V. Eberhard, Zur Morphologie der Polyeder, Teubner, Leipzig, 1891.

32. H. G. Eggleston, Convexity, Cambridge University Press, New York, 1958.

33. ———, Problems in Euclidean Space: Application of Convexity, Pergamon, New York, 1957.

34. K. Fan, Convex Sets and their Applications, Lecture Notes, Argonne National Laboratory, Illinois, 1959.

35. L. Fejes Tóth, Lagerungen in der Ebene, auf der Kugel und im Raum, Springer, Berlin, 1953.

36. ———, Regular Figures, Pergamon, London, 1964.

37. W. Fenchel, ed., Convex Cones, Sets and Functions, Lecture Notes, Princeton University, 1951.

38. W. Fenchel, Proceedings of the Colloquium on Convexity, Copenhagen, Denmark, 1965, University of Copenhagen, 1967.

39. D. Gale, The Theory of Linear Economic Models, McGraw-Hill, New York, 1960.

40. A. A. Goldstein, Constructive Real Analysis, Harper and Row, New York, 1967.

41. A. Goullet de Rugy, Géométrie des Simplexes, Centre de Documentation Universitaire et S.E.D.E.S. Réunis, Paris, 1968.

42. B. Grünbaum, Borsuk's problem and related questions, in Convexity, Proc. Sympos. Pure Math., vol. 7, Amer. Math. Soc., (1963) 271–284.

43. ———, Convex Polytopes, Interscience-Wiley, London, 1967.

44. ——— and V. Klee, Convexity and Applications (L. Durst, ed.), Proc. CUPM Geometry Conference, Santa Barbara, 1967, MAA Committee on the Undergraduate Program in Mathematics, Berkeley, 1967.

45. ——— and G. C. Shepard, Convex polytopes, Bull. London Math. Soc., 1 (1969) 257–300.

46. H. Hadwiger, Altes und Neues über konvexe Körper, Birkhäuser, Basel, 1955.

47. ———, Vorlesungen über Inhalt, Oberfläche und Isoperimetrie, Springer, Berlin, 1957.

48. ———, H. Debrunner, and V. Klee, Combinatorial Geometry in the Plane, Holt, Rinehart, and Winston, New York, 1964.

49. H. Halkin, On the necessary condition for optimal control of nonlinear systems, J. Analyse Math, 12 (1964) 1–82.

50. A. D. Joffe and V. M. Tikhomirov, Duality of convex functions and extremum problems, Russian Math. Surveys, 23 no. 6 (1968) 53–124. (Translated by T. Garrett from Uspehi Mat. Nauk 23, no. 6 (1968) 51–116.)

51. S. Karlin, Mathematical Methods and Theory in Games, Programming, and Economics, vols. 1–2, McGraw-Hill, New York, 1960.

52. ——— and W. J. Studden, Tchebycheff Systems: with Applications in Analysis and Statistics, Interscience-Wiley, New York, 1966.

53. N. D. Kazarinoff, Geometric Inequalities, Random House, New York, 1961.

54. J. L. Kelley, I. Namioka, et al., Linear Topological Spaces, Van Nostrand, Princeton, N. J., 1963.

55. M. G. Kendall and P. A. P. Moran, Geometrical Probability, Hafner, New York, 1963.

56. V. Klee, Some characterizations of convex polyhedra, Acta Math., 102 (1959) 79–107.

57. ———, ed., Convexity, Proc. Symp. Pure Math., Amer. Math. Soc., 7 (1963).

58. ———, Review of a Paper by W. Rudin and K. T. Smith, Math. Rev., 23A (1962) 376–377.

59. ———, Convex polytopes and linear programming, Proceedings of the IBM Scientific Computing Symposium on Combinatorial Problems, IBM Data Processing Division, White Plains, N. Y., 1966, pp. 123–158.

60. ———, Can a plane convex body have two equichordal points? this MONTHLY, 76 (1969) 54–55.

61. ———, What is the expected volume of a simplex whose vertices are chosen at random from a given convex body? this MONTHLY, 76 (1969) 286–288.

62. ———, Separation and support properties of convex sets—a survey, Control Theory and the Calculus of Variations, Academic Press, New York, 1969, pp. 235–303.

63. ———, Convexité, Encyclopaedia Universalis, Paris, 4 (1970) 982–985.

64. G. Köthe, Topological Vector Spaces I, Springer, New York, 1969. (Translated from the 1965 German edition by D. J. H. Garling.)

65. M. A. Krasnoselskiĭ and Ya B. Rutickiĭ, Convex Functions and Orlicz Spaces, Noordhoff, Groningen, 1961. (Translated by L. F. Boron from the 1958 Russian edition.)

66. H. Kuhn, Approximate Search for Fixed Points, Computing Methods in Optimization Problems—2, Academic Press, New York, 1969, pp. 199–211.

67. ——— and A. W. Tucker, eds., Linear Inequalities and Related Systems, Princeton University Press, 1956.

68. J. Lindenstrauss, A short proof of Liapounoff's convexity theorem, J. Math. Mech., 15 (1966) 971–972.

69. L. A. Lyusternik, Convex Figures and Polyhedra, Dover, New York, 1963. (Translated by T. J. Smith from the 1956 Russian edition.)

70. P. McMullen, The maximum numbers of faces of a convex polytope, Mathematika, 17 (1970) 179–184.

71. H. Minkowski, Gesammelte Abhandlungen, Teubner, Berlin, 1911.

72. P. A. P. Moran, A note on recent research in geometrical probability, J. Appl. Probability, 3 (1966) 453–463.

73. ———, A second note on recent research in geometrical probability, Advances in Appl. Probability, 1 (1969) 73–89.

74. J. J. Moreau, Fonctionnelles Convexes, Lecture Notes, Séminaire "Equations aux Dérivées Partielles," College de France, 1966–67.

75. T. S. Motzkin, Beiträge zur Theorie der linearen Ungleichungen, Ph.D. Thesis, Basel, 1933 (Jerusalem, 1936).

76. R. R. Phelps, Lectures on Choquet's Theorem, Van Nostrand, Princeton, N. J., 1966.

77. V. Pogorelov, Die Verbiegung konvexer Flächen, Akademie-Verlag, Berlin, 1958. (Translated from the 1951 Russian edition.)

78. T. Popoviciu, Les Fonctions Convexes, Hermann, Paris, 1945.

79. J. R. Reay, Generalizations of a theorem of Carathéodory, Memoir No. 54, Amer. Math. Soc., 1965.

80. R. T. Rockafellar, Convex Analysis, Princeton University Press, 1970.

81. C. A. Rogers, Packing and Covering, Cambridge University Press, 1964.

82. E. Steinitz and H. Rademacher, Vorlesungen über die Theorie der Polyeder, Springer, Berlin, 1934.

83. J. Stoer and C. Witzgall, Convexity and Optimization in Finite Dimensions I, Springer, Berlin, 1970.

84. A. C. Tucker, Matrix characterizations of circular-arc graphs, Pacific J. Math., 35 (1971) to appear.

85. F. A. Valentine, Convex Sets, McGraw-Hill, New York, 1964.

86. G. Wegner, Eigenschaften der Nerven homologisch-einfacher Familien im R^n, Ph.D. Thesis, Göttingen, 1967.

87. I. M. Yaglom and V. G. Boltyanskiĭ, Convex Figures, Holt, Rinehart, and Winston, New York, 1961.

88. W. Zangwill, Nonlinear Programming: A Unified Approach, Prentice-Hall, Englewood Cliffs, N. J., 1969.

CYCLIC PURSUIT OR "THE THREE BUGS PROBLEM"

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I. Introduction. A well-known problem that keeps making the rounds is the one of three (or more) bugs pursuing each other cyclically, each traveling with the same speed and having started initially at the vertices of a regular polygon. Usually one wants to know the distance traveled by each bug until mutual capture. The problem of the three bugs can be traced back to H. Brocard [1] in 1877. However, it appears probable that a wide dissemination of the problem was first due to the problem (with four and three dogs instead of bugs) appearing in the book of Steinhaus [2] which first appeared in 1950.