BOOLEAN MATRICES AND SWITCHING NETS

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1. Introduction. The problem of synthesis of switching systems is one of ever increasing importance to modern technology; it arises in the design of all sorts of pushbutton devices. The recent appearance of a number of papers [**1**–**7**] on switching circuit theory in which boolean matrices have been used makes it appropriate to discuss certain of these applications and their significances. In the present paper, a short review of boolean matrix algebra which has arisen in switching circuit theory is given, though no attempt is made to give more than an outline of certain phases of the subject. Later, a few possible extensions are discussed.

2. Boolean matrices. A boolean matrix [**3**] is simply a matrix over a boolean algebra, i.e., a rectangular array of elements from a boolean algebra. These arrays are subject to appropriate rules of operation, some of which are analogous to the rules of operation for ordinary matrices, whereas others reflect the boolean character of the elements.

Let $\tilde{B}$ be a boolean algebra of at least two elements, and M be the set of all boolean matrices of order $m\times n$. The following notation will be used to indicate intersection, union, complementation, and inclusion: Suppose $\tilde{B}=\{a, b, c, \cdots\}$. We will write $a\cap b$ for intersection, $a\cup b$ for union, $\bar{a}$ for complementation, $a\leqq b$ for inclusion.

In the set $M=\{A, B, C, \cdots\}$ where $a_{ij}, b_{ij}, c_{ij}, \cdots, (i=1, 2, \cdots, m; j=1, 2, \cdots, n)$ are elements of corresponding matrices $A, B, C, \cdots$, respectively, we define

$$\begin{array}{lll} A = B & \text{iff } a_{ij} = b_{ij} & \text{for all } i \text{ and } j \\ A = B\cap C & \text{iff } a_{ij} = b_{ij}\cap c_{ij} & \text{for all } i \text{ and } j \\ A = B\cup C & \text{iff } a_{ij} = b_{ij}\cup c_{ij} & \text{for all } i \text{ and } j \\ A = \bar{B} & \text{iff } a_{ij} = \bar{b}_{ij} & \text{for all } i \text{ and } j \\ A \leqq B & \text{iff } a_{ij} \leqq b_{ij} & \text{for all } i \text{ and } j. \end{array}$$

If 0 and 1 are the zero and universal elements of $\tilde{B}$, then in M the zero, identity, and universal matrices O, W, and I are defined as:

$$O=[0],\quad W=[e_{ij}] \text{ where } e_{ij} = 0 \quad \text{for } i\neq j$$
$$= 1 \quad \text{for } i=j$$

and $I=[1]$ for all i and j. The transpose of A, denoted by A^t, is defined by $A^t=[d_{ij}]$ where $d_{ij}=a_{ji}$ for all i and j.

It is easy to show that under these definitions M forms a boolean algebra. A multiplication may be defined as follows:

$$AB = \left[\bigcup_{k=1}^{q} (a_{ik}\cap b_{kj})\right]_{p\times r}$$

where A is a boolean matrix of order $p \times q$ and B is of order $q \times r$. If this operation is used instead of $A \cap B \in M$ and $m=n$, then M does not form a boolean algebra with respect to the operations "$\cdot$", "$\cup$" and "$-$". However, under these definitions and the given multiplication, M forms a semi-group with identity element.

In analogy to ordinary matrix theory, Luce [3] has defined concepts of symmetry, skew symmetry, orthogonal and inverse of a boolean matrix. A boolean matrix A is said to be *symmetric* iff $A^t \cap \overline{A} = O$; A is *skew-symmetric* iff $A^t \cap A = O$; the *inverse* of A, if it exists, denoted by A^{-1}, is such that $AA^{-1} = A^{-1}A = W$; and A is *orthogonal* iff it has an inverse which is A^t. With these definitions some interesting properties are found. Those discovered by Luce [3] will be restated as follows without proof.

THEOREM 1. *A boolean matrix has an inverse iff it is orthogonal.*

THEOREM 2. *Any boolean matrix can be uniquely decomposed into the disjoint union of a symmetric and a skew-symmetric matrix.*

Two observations could be given here for these definitions and theorems formulated by Luce which will be stated as Corollary 1 and Corollary 2.

COROLLARY 1. *A boolean matrix A is symmetric iff $A = A^t$ or $A^t \cup \overline{A} = \mathrm{I}$.*

COROLLARY 2. *A is involutory ($AA = W$) iff A is both orthogonal and symmetric.*

In addition to those in the literature [3, 6, 8] yet other definitions, parallel to one in ordinary matrix theory, may be considered. Since these new terms do lead to some interesting results, their existence is therefore justified.

DEFINITION 1. *A boolean matrix A is said to be tranjugate iff $A^t \cup A = \mathrm{I}$.*

LEMMA 1. *Given any $A, B \in M$, then $A \cup B = \mathrm{I}$ iff $\overline{A} \leqq B$.*

COROLLARY 3. *A boolean matrix A is tranjugate iff $\overline{A}^t \leqq A$.*

COROLLARY 4. *If a boolean matrix A is tranjugate, then the diagonal elements must be* 1.

These results are especially useful since a switching matrix has been defined [6] as any boolean matrix having all 1's down the main diagonal whereas in a boolean matrix A the elements down the main diagonal are arbitrary over $\tilde{B}$. For a skew-symmetric matrix the elements along the main diagonal are 0 since $a_{ii} \cap a_{ii} = 0$ for all i.

THEOREM 3. *Any boolean matrix can be uniquely decomposed into the joint intersection of a symmetric and a tranjugate matrix.*

Proof. For any decomposition $A = S \cap T$ subject to $S^t \cap \overline{S} = O$, $T^t \cup T = I$, and $S \cup T = I$, one has

$$\begin{aligned} A \cup A^t &= (S \cap T) \cup (S \cap T)^t = ((S \cap T) \cup S^t) \cap ((S \cap T) \cup T^t) \\ &= (S \cup S^t) \cap (S^t \cup T) \cap (S \cup T^t) \cap (T \cup T^t) = I \cap S = S \end{aligned}$$

$$\begin{aligned} A \cup \bar{A}^t &= (S \cap T) \cup (\bar{S}^t \cup \bar{T}^t) = (S \cup \bar{S}^t \cup \bar{T}^t) \cap (T \cup \bar{S}^t \cup \bar{T}^t) \\ &= I \cap (T \cup \bar{S} \cup \bar{T}^t) \\ &= T \cup \bar{S} && \text{(by Corollary 3)} \\ &= T && (\text{since } S \cup T = I \text{ iff } \bar{S} \leqq T, \text{ implies } \bar{S} \cup T = T) \end{aligned}$$

so that at most one such decomposition can exist.

Moreover, the choices of S, T suggested by the uniqueness argument above do in fact always give a decomposition of the required type. For, given any boolean matrix A and defining $S = A \cup A^t$ and $T = A \cup \bar{A}^t$ one obtains

$$\begin{aligned} A &= A \cup (A^t \cap \bar{A}^t) = (A \cup A^t) \cap (A \cup \bar{A}^t) = S \cap T \\ S^t \cap \bar{S} &= (A \cup A^t) \cap (\bar{A} \cap \bar{A}^t) \\ &= (A \cap \bar{A} \cap \bar{A}^t) \cup (A^t \cap \bar{A} \cap \bar{A}^t) = 0 \\ T^t \cup T &= (A^t \cup \bar{A}) \cup (A \cup \bar{A}^t) = A \cup \bar{A} \cup A^t \cup \bar{A}^t = I \\ S \cup T &= I. \end{aligned}$$

This completes the proof of the theorem.

3. Switching matrices and matrices associated with combinational circuits. As mentioned in the previous section, any boolean matrix which has all 1's down the main diagonal will be called a switching matrix. A switching circuit with p output terminals is said to be *combinational* if the outputs are a unique function of the inputs.

From an abstract point of view, any switching circuit can be represented by a weighted, directed, linear graph (or simply digraph). A *directed linear graph G* consists of a set V of elements called nodes together with a set E of ordered pairs of the form (i, j), i and $j \in V$, called the edges of the graph; the node i is called the *initial node* and node j the *terminal node.* A directed graph S is a *subgraph* of G if the nodes and edges of S are nodes and edges respectively of G and if each edge of S has the same initial node and the same terminal node in S as in G. If A is a subset of V, the *sectional graph* $G[A]$ of G defined by A is the subgraph whose node set is A and whose edges are all those edges in G which connect two nodes in A. When $A = V$ the sectional graph is G itself. A *directed path* p_{ij} is a subgraph of the form $p_{ij} = (i, k_1)(k_1, k_2)(k_2, k_3) \cdots (k_m, j)$, where i, j and k_t (for $t = 1, \cdots, m$) are nodes in V. It is required that all the nodes of p_{ij} shall be distinct.

In a combinational circuit, after a brief operate-time, the state f_{ij} of the "connection" between the nodes i and j depends only on the combination of values assumed by the input variables, and hence may be represented as a boolean function. In order to describe the terminal behavior of the digraph associated with such a circuit, two matrices are defined [6] as follows:

$$\text{Output matrix } F = [f_{ij}]_{p \times p}$$

where f_{ij} is the boolean function between nodes i and j, and p is the number of the output terminals. Since a node is always connected to itself, f_{ii} is defined to be 1 for each i.

$$\text{(Primitive) Connection matrix } C = [c_{ij}]_{n\times n} \qquad (c_{ii} = 1 \text{ for each } i),$$

where c_{ij} is the boolean function associated with the edge (i, j), $i \neq j$, of the corresponding digraph and n is the number of nodes in the digraph. Nodes associated with output terminals are called *output nodes*; otherwise, *nonoutput nodes*.

If the boolean algebra $\tilde{B}$ contains only two elements, denoted by 0 and 1, then c_{ij} represents the "connection" between nodes i and j. This symbol has the value 0 if there is no connection at all and 1 if there is a short circuit, but otherwise it is the symbol denoting a single contact or a union of such symbols. Properties and techniques discussed here are more general and can be applied to any boolean algebra.

Certainly any switching matrix may be interpreted as a connection matrix of a combinational switching circuit or a digraph. However, the question of how to characterize an output matrix is more interesting, and the answer is given by Hohn and Schissler [6].

THEOREM 4. *The necessary and sufficient condition that a switching matrix F be an output matrix is that $F^2 = F$.*

Another useful tool in the analysis of switching circuits is the or-determinant [2] of a square boolean matrix A of order m, the row expansion of which is defined as

$$\det A = \bigcup_{(j)} a_{1j_1}a_{2j_2} \cdots a_{mj_m}$$

where the symbol (j) means that the union is to be extended over all permutations $j_1j_2 \cdots j_m$ of the integers $1, 2, \cdots, m$.

Similarly, the *cofactor* of a_{ij}, denoted by $\det A_{ij}$, of $\det A$ is defined as the or-determinant of the matrix obtained from A by striking out the ith row and jth column of A. The *adjoint matrix of* A, denoted by $\{A\}$, is the matrix $[d_{ij}]$ of order m such that $d_{ij} = \det A_{ji}$; for the present case $a_{ii} = 1$ for all i.

These definitions are like the definitions of the determinant of a matrix over the complex field except that "$\cup$" replaces "$+$" and no sign variations appear. From this it follows that the purely combinational properties of determinants of matrices over the complex field will also apply in the boolean case. Those properties which depend on the signs of the terms may not be carried over in the same way, however; e.g.,

1. $\det A^t = \det A$.
2. The interchange of two lines of a matrix leaves the determinant invariant.
3. If A_i, $i = 1, 2, \cdots, m$ are the columns of A, then

$$\det [A_1, A_2, \cdots, \beta A_k, \cdots, A_m] = \beta \det A$$

$$\det [A_1 \cup A_i, A_2, \cdots, A_m] = (\det A) \cup (\det [A_i, A_2, \cdots, A_m])$$

where β is a scalar.

4. Laplace expansion also holds in this case except there are no sign variations.

For the use in the analysis of switching circuits the following result is obtained.

THEOREM 5. *If C is the connection matrix of a switching circuit, and F is the output matrix, then $F=\{C\}$.*

4. The analysis and synthesis of combinational circuits. The basic problem of analysis of combinational circuits is the determination of the relation between any given connection matrix and the corresponding output matrix. To accomplish this, Hohn and Schissler [6] showed first how to obtain from a given circuit an equivalent circuit using one less non-output node in the formation of the connection matrix. This operation (the star-mesh transformation [6]) is repeated until there are no non-output nodes in the accounting.

Matrixwise, this operation proceeds as follows: To remove a non-output node r, one joins to entry c_{ij} of C the intersection of the entry c_{ir} by the entry c_{rj}, thereafter deleting row r and column r from C. It is interesting to note this node removal process is very similar to Chio's method for the reduction of the order of a determinant in ordinary matrix theory. This method is very useful because it can be reversed, i.e., node insertion. In other words if a new line is to be placed in a reduced matrix, every row and every column must contain a same element, and the intersection of these two elements must in turn be contained in one entry.

A natural question to ask at this point is, "Is it possible to generalize the single-node removal process to the process of multiple-node removal?" Shekel [8] in an unpublished note first obtained this generalization in an algebraic form. However, Shekel's process can be accomplished topologically as follows.

Let C be the connection matrix (not necessarily symmetric) of a given digraph G; let C be partitioned in such a way that

$$C = [c_{ij}]_{n\times n} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}_{n\times n}$$

where C_{11} is a square submatrix of order p which corresponds to the output nodes of G.

For convenience, a mapping function f is defined from the edges (i, j) of G into a boolean algebra $\tilde{B}$ such that $f((i, j)) = c_{ij}$ for all edges in G where $c_{ij}\in\tilde{B}$. This definition is extended to any nonempty subgraph R of G as follows:

$$f(R) = \bigcap f((u_1, u_2))$$

where the intersection runs over all edges (u_1, u_2) in R.

Next let

$$C_{t_1t_2} = [c_{ij}^{t_1t_2}] \quad t_1, t_2 = 1, 2, \qquad \{C_{22}\} = [m_{ij}]_{(n-p)\times(n-p)}$$

and

$$F = [f_{ij}]_{p\times p}$$

where $\{C_{22}\}$ is the adjoint matrix of C_{22}.

Yoeli [8] has shown that if F is the corresponding output matrix of order p, then $F=\{C_{11}\cup C_{12}\{C_{22}\}C_{21}\}$. Suppose $C_{11}\cup C_{12}\{C_{22}\}C_{21}=[d_{ij}]$, then

$$d_{ij} = \overset{11}{c_{ij}} \cup \left[\bigcup_{k,t=1}^{n-p} (\overset{12}{c_{ik}} \cap m_{kt} \cap \overset{21}{c_{tj}}) \right].$$

Since m_{kt} is the switching function from node k to node t of the circuit corresponding to C_{22}, it follows that

$$m_{kt} = \bigcup_{p'_{kt}} f(p'_{kt})$$

where p'_{kt} is a directed path from node k to node t in the sectional graph $G[A]$ where A is the node set corresponding to the submatrix C_{22}, and the union is taken over all possible $p'_{kt}\in G[A]$. Therefore

$$\begin{aligned} d_{ij} &= \bigcup_{p_{ij}} f(p_{ij}) && \text{for } i \neq j \\ &= 1 && \text{for } i = j \end{aligned}$$

where p_{ij} is a directed path from node i to node j in the sectional graph $G[A\cup i\cup j]$, where $A\cup i\cup j$ is the set union of the nodes i, j, and the node set A; and the union is taken over all possible $p_{ij}\in G[A\cup i\cup j]$. One may now state

THEOREM 6. *Let V be the node set of a given digraph G, and V_m be any subset of the node set which corresponds to the non-output nodes of V. Then the digraph G_r and G have the same output matrix where G_r is derived from G by the following procedure.*

(a) *Remove $G[V_m]$ from G, i.e., remove all nodes in V_m and also all edges incident to and coming from any node in V_m. Edges which do not previously exist in G may be considered as edges which map to zero, i.e., $f((i, j))=0$ for all (i, j) not in G where f is the mapping function of G.*

(b)
$$\begin{aligned} f_r((i,j)) &= \bigcup_{p_{ij}} f(p_{ij}) && \text{for } i \neq j \\ &= 1 && \text{for } i = j \end{aligned}$$

for all $i, j\in(V-V_m)$ where f_r is the mapping function of G_r; $(V-V_m)$ represents the elements contained in V but not in V_m; p_{ij} is a direct path from nodes i to j in the sectional graph $G[V_m\cup i\cup j]$ where $V_m\cup i\cup j$ is the set union of the nodes i, j and the node set V_m; and the union is taken over all possible $p_{ij}\in G[V_m\cup i\cup j]$.

It is interesting to note that if V_m contains only one node, the above theorem reduces to the star-mesh transformation in Hohn and Schissler's paper [6].

At this point, it is obvious that the topological reduction process not only displays in a very intuitive manner the causal relationships among the variables of the system under study, but also shows that the process is independent of the labelling of the nodes.

Example 1. Consider the circuit shown in Fig. 1(a). The dotted part is the sectional graph to be removed. The reduced circuit is shown in Fig. 1(b). (For convenience, "$\cup$," "$\cap$," and "$-$" are replaced by "$+$" juxtaposition, and "$'$" in all the figures, respectively.)

Evidently this process can be easily applied to sequential circuits to give the corresponding multiple "state removal" algorithm with minor modifications.

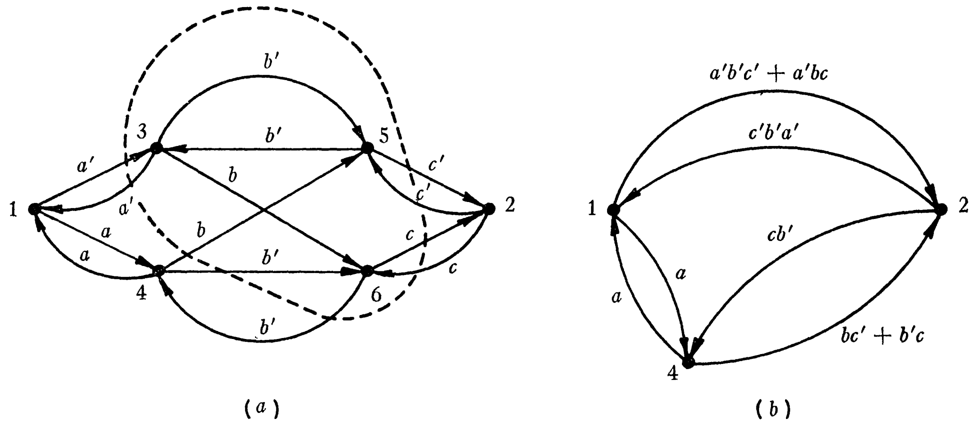

FIG. 1 (a) Example for multiple-node removal.
(b) The corresponding reduced circuit of (a).

If $G[A]$ is the sectional graph to be removed from a state diagram G where A is a set of states and if there exist self-loops in $G[A]$, then the process shown in Fig. 2 must be used in order to eliminate all such self-loops (1 is used as identity for multiplication but $1 \cup b \neq 1$). k is a nonnegative integer.

After all the self-loops having been removed, Theorem 6 now can be applied to obtain the corresponding reduced state diagram.

Example 2. Consider the state diagram of Fig. 3(a). The dotted part is the sectional graph to be removed. Fig. 3(b) is the corresponding reduced state diagram.

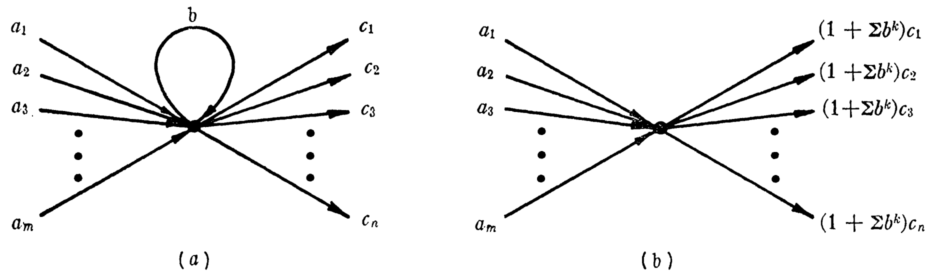

FIG. 2 (a) A node with a self-loop.
(b) With the self-loop removed.

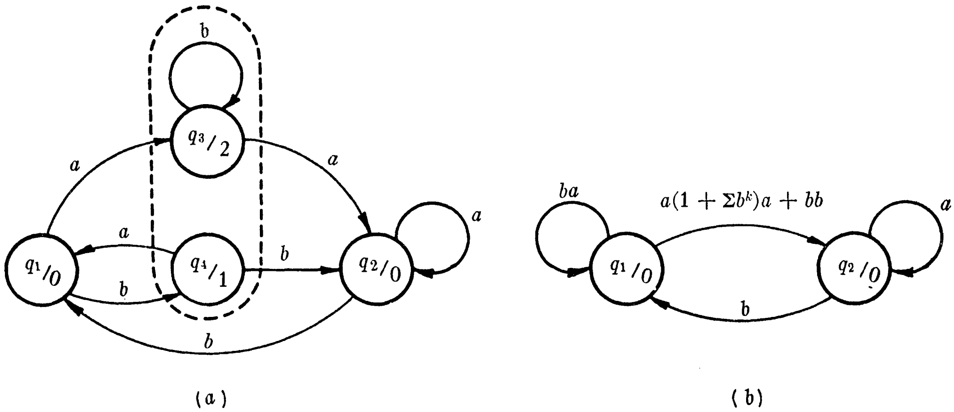

FIG. 3. Example for multiple state removal.

5. Conclusions. In this paper the basic properties of boolean matrices have been discussed and their applications to the analysis of both combinational relay and sequential circuits were studied. As mentioned earlier, no attempt is made to give more than an outline of certain phases of the subject. For a more extensive treatment, reference should be made to the various papers and works in the field.

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