A Tail of Two Palindromes

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> It was the best of times, it was the worst of times, it was the age of wisdom, it was the age of foolishness, it was the epoc of belief, it was the epoc of incredulity, it was the season of light, it was the season of darkness, it was the spring of hope, it was the winter of despair, we had everything before us, we had nothing before us, we were all going direct to heaven, we were all going direct the other way—in short, the **period** was so far like the present **period**. . . .
>
> —Charles Dickens

1. A TALE OF TWO CONJUGATES $\big[\,\ldots,\overline{\text{Vanna, wanna ‘V’?; “I prefer pi.”}}\,\big]$**.** Upon a preliminary perusal, this parable may appear to be about pairs of palindromes, periods, and pitiful alliteration. In actuality, however, it is the story of a real quadratic irrational number α and its long-lost younger sibling, its algebraic conjugate $\widetilde{\alpha}$ ($\alpha > \widetilde{\alpha}$). How in the dickens are all these notions connected? We begin at the beginning. . . .

Although the conjugates α and $\widetilde{\alpha}$ are not *identical* twins, unlike the two zeros of $(x-3)^2$, they do share a common family history: they each were born of the same irreducible parent polynomial having rational coefficients,

$$P_\alpha(x) = P_{\widetilde{\alpha}}(x) = (x-\alpha)(x-\widetilde{\alpha}) = x^2 - \mathrm{Trace}(\alpha)x + \mathrm{Norm}(\alpha),$$

where $\mathrm{Trace}(\alpha) = \alpha + \widetilde{\alpha}$ and $\mathrm{Norm}(\alpha) = \alpha\,\widetilde{\alpha}$. Perhaps not surprisingly, some conjugate pairs exhibit similar personalities. But how similar can they be? And how can we detect those similarities simply by looking at α? As we will discover as our tale unfolds, the answer—foreshadowed in the title—is encoded in what can be described as the number theoretic analogue of the DNA-sequence for α. However, before delving into α's genes, we first motivate our main results by weaving a lattice of algebra.

The setting for our story is an ordinary real quadratic field $\mathbb{Q}(\sqrt{d}\,)$ for some squarefree positive integer d. No matter whether one views $\mathbb{Q}(\sqrt{d}\,)$ as a familiar field or a foreign one, all we require here is the basic observation that

$$\mathbb{Q}(\sqrt{d}\,) = \mathbb{Q} + \mathbb{Q}\sqrt{d} = \big\{r + s\sqrt{d} : r, s \in \mathbb{Q}\big\}.$$

Given α in $\mathbb{Q}(\sqrt{d}\,)$, we define the *lattice generated by* α, denoted $\mathbb{Z} + \mathbb{Z}\alpha$, by

$$\mathbb{Z} + \mathbb{Z}\alpha = \{m + n\alpha : m, n \in \mathbb{Z}\}.$$

Those readers who are algebraically inclined will recognize $\mathbb{Z} + \mathbb{Z}\alpha$ as a $\mathbb{Z}$-*module* in $\mathbb{Q}(\sqrt{d}\,)$, namely, the $\mathbb{Z}$-module generated by 1 and α. In order to set the scene, we note that it is a relatively easy matter to determine when two lattices are identical. In fact, the story line behind the short proof is so straightforward that we suppress it here.

Proposition 1. *For irrational numbers α and β in $\mathbb{Q}(\sqrt{d}\,)$, the lattices $\mathbb{Z} + \mathbb{Z}\alpha$ and $\mathbb{Z} + \mathbb{Z}\beta$ are identical if and only if either $\alpha + \beta$ or $\alpha - \beta$ is an integer.*

If we now think geometrically, then we can consider two lattices in $\mathbb{Q}(\sqrt{d}\,)$ to be "similar" if one is a "scalar multiple" of the other. More precisely, we say that the lattices $\mathbb{Z}+\mathbb{Z}\alpha$ and $\mathbb{Z}+\mathbb{Z}\beta$ in $\mathbb{Q}(\sqrt{d}\,)$ are *similar*, expressed symbolically by $\mathbb{Z}+\mathbb{Z}\alpha \approx \mathbb{Z}+\mathbb{Z}\beta$, if there exists γ in $\mathbb{Q}(\sqrt{d}\,)$ such that $\mathbb{Z}+\mathbb{Z}\alpha = \gamma(\mathbb{Z}+\mathbb{Z}\beta)$. While the notion of similar modules is fundamental in classical algebraic number theory, prompted by Proposition 1 we focus here on the natural question begging to be asked: When are two lattices similar? The answer comes into focus through the following not-widely-known mathematical vignette:

Proposition 2. *For α and β in $\mathbb{Q}(\sqrt{d}\,)$, the lattices $\mathbb{Z}+\mathbb{Z}\alpha$ and $\mathbb{Z}+\mathbb{Z}\beta$ are similar if and only if there exist integers A, B, C, and D satisfying $AD-BC=\pm 1$ such that*

$$\alpha = \frac{A+B\beta}{C+D\beta}. \tag{1}$$

A sketchy "Cliff Notes" version of the proof. If we assume that the two lattices are similar, then there exists γ in $\mathbb{Q}(\sqrt{d}\,)$ for which

$$\mathbb{Z}+\mathbb{Z}\alpha = \gamma(\mathbb{Z}+\mathbb{Z}\beta). \tag{2}$$

Thus we can find integers A, B, C, and D such that $\alpha = \gamma(A+B\beta)$ and $1 = \gamma(C+D\beta)$. After the profound insight that $\alpha = \alpha/1$, we discover that

$$\alpha = \frac{A+B\beta}{C+D\beta}.$$

Similarly, by again appealing to (2) we see that there exist integers A', B', C', and D' that produce the identities $\gamma\beta = A'+B'\alpha$ and $\gamma = C'+D'\alpha$, whence

$$\beta = \frac{\gamma\beta}{\gamma} = \frac{A'+B'\alpha}{C'+D'\alpha}.$$

A little linear algebra reveals that

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

and since all the entries in these two matrices are *integers*, we conclude that the determinant $AD-BC$ must be 1 or -1.

Conversely, we now suppose that (1) holds with integers A, B, C, and D satisfying $AD-BC=\pm 1$. If we select $\gamma = (C+D\beta)^{-1}$, then everyday algebra allows us to verify easily that

$$m+n\alpha = \gamma\big((nA+mC)+(nB+mD)\beta\big) \in \gamma(\mathbb{Z}+\mathbb{Z}\beta)$$

holds for any $m+n\alpha$ in $\mathbb{Z}+\mathbb{Z}\alpha$ and that

$$\gamma(m+n\beta) = \mp(mB-nA) \mp (-mD+nC)\alpha \in \mathbb{Z}+\mathbb{Z}\alpha$$

is also true for each $m+n\beta$ in $\mathbb{Z}+\mathbb{Z}\beta$. Thus we see that $\mathbb{Z}+\mathbb{Z}\alpha = \gamma(\mathbb{Z}+\mathbb{Z}\beta)$ and conclude that the lattices $\mathbb{Z}+\mathbb{Z}\alpha$ and $\mathbb{Z}+\mathbb{Z}\beta$ are similar, which brings our sketchy overview to a close. ∎

Many readers will recognize the fundamental relationship in (1), for it takes on supporting roles throughout nearly all branches of mathematics. In fact, two real numbers α and β are said to be *equivalent*, denoted $\alpha \sim \beta$, if there exist integers A, B, C, and D satisfying

$$\alpha = \frac{A + B\beta}{C + D\beta} \qquad (AD - BC = \pm 1); \tag{3}$$

that is, using some highbrow language, if α is a *linear fractional transformation* or a *Mellon transformation* or a *Möbius transformation* of β.

Returning now to our main quadratic characters α and $\widetilde{\alpha}$, we see by Proposition 1 that $\mathbb{Z} + \mathbb{Z}\alpha = \mathbb{Z} + \mathbb{Z}(\alpha + k)$ for any integer k. Therefore, without loss of generality, we may assume for the purpose of our story that the quadratic irrational α is greater than 1. Since $\alpha - \widetilde{\alpha}$ is irrational, Proposition 1 also reveals that the lattices $\mathbb{Z} + \mathbb{Z}\alpha$ and $\mathbb{Z} + \mathbb{Z}\widetilde{\alpha}$ are identical if and only if Trace(α) is an integer—an easy circumstance to check. On the other hand, from Proposition 2 we learn that $\mathbb{Z} + \mathbb{Z}\alpha$ and $\mathbb{Z} + \mathbb{Z}\widetilde{\alpha}$ are similar if and only if α and $\widetilde{\alpha}$ are equivalent—a far more cryptic condition to verify quickly.

One illuminating and three obscuring examples in $\mathbb{Q}(\sqrt{5})$. If $\alpha_1 = \frac{1}{2} + \sqrt{5}$, then we immediately see that Trace$(\alpha_1) = 1$ and instantly conclude that

$$\mathbb{Z} + \mathbb{Z}\left(\tfrac{1}{2} + \sqrt{5}\right) = \mathbb{Z} + \mathbb{Z}\left(\tfrac{1}{2} - \sqrt{5}\right).$$

However our story would have a dramatically different ending if our main character were $\alpha_2 = \frac{1}{3} + \sqrt{5}$ or $\alpha_3 = \frac{1}{4} + \sqrt{5}$ or $\alpha_4 = \frac{1}{5} + \sqrt{5}$. Clearly we see that Trace$(\alpha_2) = \frac{2}{3}$, Trace$(\alpha_3) = \frac{1}{2}$, and Trace$(\alpha_4) = \frac{2}{5}$ are not integers, from which we infer that $\mathbb{Z} + \mathbb{Z}\alpha_2 \neq \mathbb{Z} + \mathbb{Z}\widetilde{\alpha_2}$, $\mathbb{Z} + \mathbb{Z}\alpha_3 \neq \mathbb{Z} + \mathbb{Z}\widetilde{\alpha_3}$, and $\mathbb{Z} + \mathbb{Z}\alpha_4 \neq \mathbb{Z} + \mathbb{Z}\widetilde{\alpha_4}$. However perhaps some of those pairs of "conjugate lattices" might at least be similar.

As we discover in the next chapter of our tale, $\mathbb{Z} + \mathbb{Z}\alpha_2 \not\approx \mathbb{Z} + \mathbb{Z}\widetilde{\alpha_2}$, whereas $\mathbb{Z} + \mathbb{Z}\alpha_3 \approx \mathbb{Z} + \mathbb{Z}\widetilde{\alpha_3}$ and $\mathbb{Z} + \mathbb{Z}\alpha_4 \approx \mathbb{Z} + \mathbb{Z}\widetilde{\alpha_4}$; or stated differently, $\frac{1}{3} + \sqrt{5}$ is *not* equivalent to $\frac{1}{3} - \sqrt{5}$, whereas both $\frac{1}{4} + \sqrt{5}$ and $\frac{1}{5} + \sqrt{5}$ *are* equivalent to their conjugate younger siblings $\frac{1}{4} - \sqrt{5}$ and $\frac{1}{5} - \sqrt{5}$, respectively. Specifically we have the following highly nonobvious identities:

$$\frac{1}{4} + \sqrt{5} = \frac{445 - 179(\frac{1}{4} - \sqrt{5}\,)}{179 - 72(\frac{1}{4} - \sqrt{5}\,)},$$

where $(445)(-72) - (-179)(179) = 1$, and

$$\frac{1}{5} + \sqrt{5} = \frac{1810 - 743(\frac{1}{5} - \sqrt{5}\,)}{743 - 305(\frac{1}{5} - \sqrt{5}\,)},$$

where $(1810)(-305) - (-743)(743) = -1$. Moreover, in these two cases, the indicated coefficients are the most petite ones that satisfy the conditions of (3); that is, all four of the coefficients in each case have the smallest possible absolute values. How would someone find those somewhat hefty integers or, as in the case of $\frac{1}{3} + \sqrt{5}$, conclusively show that no such integers of any girth exist? We will soon see that if we look at α in the right way, we do not have to find those integers—when they exist, they will find us.

This little lattice conjugate conundrum inspires the following question, which in this paper we pose, answer, and place in context: *Given a real quadratic irrational* *number α, how can we simply look at it and immediately determine whether α is equivalent to its conjugate?*

2. A TALE OF TWO THEOREMS $\left[\ldots, \overline{\text{Never odd or even?; } i!}\right]$**.** For any real irrational number α we write $\alpha = a_0 + \alpha_0$, where $a_0 = [\alpha]$ signifies the *integer part of* α (a_0 is the largest integer that does not exceed α), while $\alpha_0 = \{\alpha\}$ plays the minor role known as the *fractional part of* α. Since α is irrational, we have $1/\alpha_0 > 1$, so after a dramatic double flip we see that

$$\alpha = a_0 + \frac{1}{1/\alpha_0} = a_0 + \frac{1}{a_1 + \alpha_1},$$

where $a_1 = [1/\alpha_0]$ and $\alpha_1 = \{1/\alpha_0\}$. Again, since α is irrational, we can repeat this dizzying act forever and thereby discover that

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots}}}, \tag{4}$$

where all the a_n are integers and $a_n > 0$ when $n > 0$. The expression in (4) is known as the *continued fraction expansion of* α, which we abbreviate by writing $\alpha = [a_0, a_1, a_2, \ldots]$. It is this continued fraction expansion that can be thought of as the number theoretic DNA of a real number.

By a *word W of length L*, we mean a finite string of L positive integers, $W = (w_1, w_2, \ldots, w_L)$. Reminiscent of our carefree days frolicking with repeating decimal expansions, here we write $\left[a_0, a_1, \ldots, a_M, \overline{W}\right]$ to indicate that a continued fraction expansion becomes periodic after a_M. That is, we have[1]

$$\begin{aligned}\left[a_0, a_1, \ldots, a_M, \overline{W}\right] &= \left[a_0, a_1, \ldots, a_M, \overline{(w_1, w_2, \ldots, w_L)}\right] \\ &= \left[a_0, a_1, \ldots, a_M, w_1, w_2, \ldots, w_L, w_1, w_2, \ldots, w_L, \ldots\right].\end{aligned}$$

As an illustration, we remark that any computer we chose to interrogate would quickly confess that

$$\begin{aligned}\tfrac{1}{2} + \sqrt{5} &= \left[2, \overline{(1, 2, 1, 3)}\right], \\ \tfrac{1}{3} + \sqrt{5} &= \left[2, \overline{(1, 1, 3, 9, 1, 3)}\right], \\ \tfrac{1}{4} + \sqrt{5} &= \left[2, \overline{(2, 17, 2, 3, 1, 70, 1, 3)}\right],\end{aligned}$$

and would even admit that

$$\begin{aligned}&\tfrac{1}{5} + \sqrt{5} \\ &= \left[2, \overline{(2, 3, 2, 2, 3, 2, 4, 27, 1, 2, 1, 1, 1, 3, 1, 5, 9, 1, 110, 1, 9, 5, 1, 3, 1, 1, 1, 2, 1, 27, 4)}\right].\end{aligned}$$

[1]We remark—and any continued fraction connoisseur would agree—that the parentheses surrounding the period are superfluous and nonstandard. We include them here for clarity, and we place this comment here since every literary work is required to contain at least one footnote.

As we will soon discover, just by glancing at the foregoing continued fractions we can report without missing a beat that

$$\mathbb{Z}+\mathbb{Z}\left(\tfrac{1}{2}+\sqrt{5}\right)=\mathbb{Z}+\mathbb{Z}\left(\tfrac{1}{2}-\sqrt{5}\right)$$

and

$$\mathbb{Z}+\mathbb{Z}\left(\tfrac{1}{3}+\sqrt{5}\right)\not\approx\mathbb{Z}+\mathbb{Z}\left(\tfrac{1}{3}-\sqrt{5}\right),$$

while $\mathbb{Z}+\mathbb{Z}(\frac{1}{4}+\sqrt{5})$ and $\mathbb{Z}+\mathbb{Z}(\frac{1}{5}+\sqrt{5})$ are *not* the same as but *are* similar to their conjugate lattices $\mathbb{Z}+\mathbb{Z}(\frac{1}{4}-\sqrt{5})$ and $\mathbb{Z}+\mathbb{Z}(\frac{1}{5}-\sqrt{5})$, respectively. What is the secret? The surprising answer is that we should employ some *drawkcab* thinking.

We say that a word $W=(w_1,w_2,\ldots,w_L)$ is a *palindrome* if it reads the same foward and backward; that is, if $(w_1,w_2,\ldots,w_L)=(w_L,w_{L-1},\ldots,w_1)$. Anticipating the denouement of our narrative, we recall a well-known result that provides a necessary and sufficient condition for a real quadratic irrational α (greater than 1) to have $\mathrm{Trace}(\alpha)=0$: for any rational number r greater than 1 for which $\alpha=\sqrt{r}$ is irrational, it is the case that

$$\sqrt{r}=\left[a,\overline{(W,2a)}\right] \tag{5}$$

for some positive integer a and palindrome W; conversely, for any positive integer a and palindrome W there exists a rational number r greater than 1 such that (5) holds. If $\alpha=\sqrt{r}$ with $r>1$, then we make the trivial observation that $\alpha-\widetilde{\alpha}=\sqrt{r}-(-\sqrt{r})=2\sqrt{r}>2$. We now reveal our first result, which generalizes the previous classic characterization.

Theorem 1. *Let α be a real quadratic irrational satisfying the conditions $\alpha>1$ and $\alpha-\widetilde{\alpha}>2$. Then* $\mathrm{Trace}(\alpha)$ *is an integer if and only if the continued fraction expansion of α has the form $\alpha=\left[a,\overline{(W,b)}\right]$, where a and b are positive integers and the word W is a palindrome. Moreover, in this situation,* $\mathrm{Trace}(\alpha)=2a-b$.

Thus by simply studying the structure of the continued fraction of $\alpha=\frac{1}{2}+\sqrt{5}$, $\left[2,\overline{((1,2,1),3)}\right]$, we see that its trace is integral. Moreover, we confirm that $\mathrm{Trace}(\alpha)=1$ since Theorem 1 implies that $\mathrm{Trace}(\alpha)=4-3$. We postpone the proofs of Theorem 1 and the related result to follow until chapter 4, hopefully generating some mathematical suspense in the process.

We say that the real quadratic irrational α has a *tail of two palindromes* if its continued fraction expansion takes the form

$$\alpha=\left[a_0,a_1,\ldots,a_M,\overline{(W_1,W_2)}\right],$$

where the words W_1 and W_2 are *both* palindromes. So each of the "continued fractions" following our chapter titles has a tail of two palindromes. We also note that the periodic word (W,b) in the continued fraction appearing in Theorem 1 as well as the periodic word that follows the present chapter's title are degenerate examples of tails of two palindromes, since the one-letter words "b" and "i" are each trivial palindromes. We now arrive at the result that answers the question we posed at the close of the previous chapter.

Theorem 2. *A real quadratic irrational α is equivalent to its conjugate $\widetilde{\alpha}$ if and only if α has a tail of two palindromes.*

Thus, since the period $((2, 17, 2),\ (3, 1, 70, 1, 3))$ of $\frac{1}{4} + \sqrt{5}$, and the period

$$\big((2, 3, 2, 2, 3, 2),\ (4, 27, 1, 2, 1, 1, 1, 3, 1, 5, 9, 1, 110, 1, 9, 5, 1, 3, 1, 1, 1, 2, 1, 27, 4)\big)$$

of $\frac{1}{5} + \sqrt{5}$ are both words composed of two palindromes, Theorem 2 implies that $\frac{1}{4} + \sqrt{5} \sim \frac{1}{4} - \sqrt{5}$ and $\frac{1}{5} + \sqrt{5} \sim \frac{1}{5} - \sqrt{5}$. However, since the period of $\frac{1}{3} + \sqrt{5}$ is $((1, 1), (3, 9, \mathbf{1}, 3))$—not a composition of two palindromes—we immediately conclude that $\frac{1}{3} + \sqrt{5} \not\sim \frac{1}{3} - \sqrt{5}$. We do note that if 9 were replaced by 1, then the period would become $(1, 1, 3)$ and the corresponding continued fraction would mutate to $\alpha = \big[2, \overline{(1, 1, 3)}\,\big]$. Invoking Theorem 1, we would conclude that the trace of α was integral (indeed, it would be 1), hence that $\mathbb{Z} + \mathbb{Z}\alpha = \mathbb{Z} + \mathbb{Z}\widetilde{\alpha}$. In fact, we would have

$$\alpha = \Big[2, \overline{(1, 1, 3)}\,\Big] = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{17}.$$

On the other hand, if the boldfaced **1** in the period of $\frac{1}{3} + \sqrt{5}$ were replaced by 9, then we would obtain the number $\beta = \big[2, \overline{((1, 1), (3, 9, 9, 3))}\,\big]$. In this case we could conclude on the basis of Theorem 2 that, while $\mathbb{Z} + \mathbb{Z}\beta \neq \mathbb{Z} + \mathbb{Z}\widetilde{\beta}$, it is true that $\beta \sim \widetilde{\beta}$. As a final aside for those who are curious, we report that this one-digit replacement yields

$$\beta = \Big[2, \overline{((1, 1), (3, 9, 9, 3))}\,\Big] = \tfrac{27}{37} + \tfrac{1}{74}\sqrt{18530}.$$

As an entertaining digression, we momentarily consider the special case in which the period of α is a word whose entries come from a two-element set, for example, $\alpha = \big[3, 4, \overline{(1, 2, 1, 2, 2, 1)}\,\big]$. Theorem 2 immediately implies that, unless $\alpha \sim \widetilde{\alpha}$, the period of such a quadratic irrational cannot be too modest in length. Specifically we have:

Corollary 1. *Suppose that $\alpha = \big[a_0, a_1, \dots, a_M, \overline{W}\,\big]$, where W is a word of length L formed from a two-integer alphabet $\{a, b\}$. If $\alpha \not\sim \widetilde{\alpha}$, then $L \geq 6$.*

3. A TALE OF TWO THEORIES $\big[\ldots, \overline{\text{A Toyota!; Race carrot or race car?}}\ \big]$.

Here we briefly shine the spotlight on two important classical topics that are prominent both in the literature and in the arguments to come: the continued fractions of real quadratic irrationals and the continued fractions of equivalent numbers. In addition, we expose the intimate relationship that exists between continued fractions and 2×2 matrices. There are many venues where the full story behind these ideas is played out (see, for example, [**1**]). Our terse overview begins with a celebrated theorem of Lagrange from 1770 [**3**].

Lagrange's Theorem. *A real number α is a quadratic irrational if and only if its continued fraction expansion is eventually periodic.*

We continue our rapid review by recalling that a real quadratic irrational α is *reduced* if $\alpha > 1$ and $-1 < \widetilde{\alpha} < 0$. It is a simple matter to show that for any given irrational α in $\mathbb{Q}(\sqrt{d}\,)$ with $\alpha - \widetilde{\alpha} > 2$ there exists an integer k such that $\alpha + k$ is reduced. But how can we determine whether a quadratic is reduced by merely gazing into its continued fraction expansion? Before dueling with pistols, Galois dueled with the less fatal proposition of continued fractions and answered this question. In fact his first published result, dating from 1829, was the following [**2**]:

Galois's Theorem. *A real quadratic irrational* α *is reduced if and only if* α *is purely periodic. Moreover, if* $\alpha = \left[\,\overline{W}\,\right]$ *with* $W = (w_1, w_2, \ldots, w_L)$*, then* $-1/\widetilde{\alpha} = \left[\,\overline{\overleftarrow{W}}\,\right]$*, where* $\overleftarrow{W} = (w_L, w_{L-1}, \ldots, w_1)$*.*

We highlight one last beautiful classic result. If we seek an example of equanimity within the context of impressionistic painting, then we might point to the well-known work of Georges Seurat. However if we seek an impressionistic view of equivalent numbers within the context of continued fractions, then we might turn to an 1847 theorem of Joseph Serret [**4**].

Serret's Theorem. *Real numbers* α *and* β *are equivalent if and only if from some point onward the tails of their continued fractions coincide.*

So, for example, Serret's theorem tells us that $\left[3, 7, 5, 8, \overline{(2, 1)}\,\right] \sim \left[9, 6, \overline{(1, 2)}\,\right]$.

Now, in a surprising turn of events, we find 2×2 matrices suddenly appearing on the scene and transforming the linear flow of our story. The star-crossed relationship between continued fractions and 2×2 matrices is dramatically portrayed in the following lemma, which can be verified by induction.

Lemma 1. *For nonzero real numbers* $c_1, c_2, \ldots, c_n$*, let* r_n *and* s_n *be defined by* $r_0 = 1$*,* $s_0 = 0$*, and*

$$\begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c_2 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} c_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} r_n & r_{n-1} \\ s_n & s_{n-1} \end{pmatrix} \tag{6}$$

for $n \geq 1$*. Then*

$$\frac{r_n}{s_n} = c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \cfrac{1}{\ddots + \cfrac{1}{c_n}}}}.$$

We caution the casual reader that the continuedesque fraction appearing in Lemma 1 is an "illegitimate" continued fraction, for the c_n might *not* be integers. Moreover, given that irrational c_n are allowed, the r_n/s_n terms might *not* be as rational as the notation would suggest. With this somewhat scandalous revelation behind us, we close the chapter with a bizarre twist: by taking the transpose of (6) we find that the word $(c_1, c_2, \ldots, c_n)$ is a *palindrome* if and only if $s_n = r_{n-1}$ (i.e., if and only if the matrix on the right in (6) is *symmetric*).

4. A TALE OF TWO PROOFS $\left[\ldots, \overline{\text{So many dynamos.; Ed is on no side.}}\,\right]$**.** We open this chapter by establishing Theorem 1.

Proof of Theorem 1. First we suppose that $\alpha = \left[a, \overline{(W, b)}\,\right]$, where a and b are positive integers and the finite string of positive integers W is a palindrome. We claim that Trace(α) is integral. If we write $W = (w_1, w_2, \ldots, w_{L-1}, w_L) = (w_1, w_2, \ldots, w_2, w_1)$

and

$$\begin{pmatrix} w_1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} w_2 & 1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} w_2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} w_1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} r & s \\ u & t \end{pmatrix},$$

then the previous chapter's final twist reveals that $s = u$. The periodic continued fraction of α allows us to declare *formally* that

$$\alpha = \left[a, W, b, \frac{1}{\alpha - a}\right],$$

where we refer to this expression as "formal" (rather than "illegitimate," as in the preceding chapter) because $1/(\alpha - a)$ is *not* an integer. As a by-product of the two previous observations and Lemma 1, we arrive at the following matrix product:

$$\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} r & s \\ s & t \end{pmatrix}\begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\alpha-a} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \alpha & \bullet \\ 1 & \bullet \end{pmatrix}.$$

After a brief dalliance involving matrix multiplication, our young irrational discovers that

$$\frac{abr + (a+b)s + t + (ar+s)(\alpha - a)}{br + s + r(\alpha - a)} = \frac{\alpha}{1},$$

which following some stormy calculations blossoms into the blissful relationship

$$r\alpha^2 - \big((2a-b)r\big)\alpha + \big((a-b)ar - bs - t\big) = 0.$$

Therefore we see that $\mathrm{Trace}(\alpha) = 2a - b$ is an integer, which brings the first episode of our proof to a close.

Next we assume that $t = \mathrm{Trace}(\alpha) = \alpha + \widetilde{\alpha}$ is an integer. We initially consider the case in which α is reduced—that is, α is purely periodic, say $\alpha = \left[\,\overline{V}\,\right]$ with $V = (v_1, v_2, \ldots, v_L)$. Invoking Galois's theorem, we can assert that $-1/\widetilde{\alpha} = \left[\,\overline{\overleftarrow{V}}\,\right]$, whence

$$\left[t, \overline{\overleftarrow{V}}\,\right] = t + \frac{1}{(-1/\widetilde{\alpha})} = t - \widetilde{\alpha} = \alpha = \left[\,\overline{\overrightarrow{V}}\,\right].$$

Thus we must have $(t, v_L, v_{L-1}, \ldots, v_2) = (v_1, v_2, \ldots, v_L)$, which implies that the continued fraction of α has the form $\alpha = \left[t, \overline{(W, t)}\,\right]$ with $W = (v_2, v_3, \ldots, v_3, v_2)$ a palindrome. This observation, together with the realization that $2t - t = t = \mathrm{Trace}(\alpha)$, establishes the result in the special case in which α is reduced.

If α is not reduced, then since $\alpha > 0$ and $\alpha - \widetilde{\alpha} > 2$, we can find an integer k such that $\alpha + k$ *is* reduced. Thus $\alpha + k = \left[\,\overline{V}\,\right]$ for some word V. Harking back to the previous case, we now know that $\alpha + k = \left[u, \overline{(W, u)}\,\right]$ for some palindrome W, where $u = \mathrm{Trace}(\alpha + k) = \mathrm{Trace}(\alpha) + 2k = t + 2k$. We conclude that

$$\alpha = \left[t + k, \overline{(W, t + 2k)}\,\right]$$

and $2(t+k) - (t+2k) = t = \mathrm{Trace}(\alpha)$, which ties up the loose ends of this short proof. ∎

We turn our attention next to Theorem 2.

Proof of Theorem 2. We begin by assuming that α has a tail of two palindromes, that is, its continued fraction expansion looks like

$$\alpha = \left[a_0, a_1, \dots, a_M, \overline{(W_1, W_2)}\right],$$

where W_1 and W_2 are palindromes. If we write α_{pp} for the reduced quadratic irrational defined by $\alpha_{\mathrm{pp}} = \left[\,\overline{(W_1, W_2)}\,\right]$, then by Serret's theorem we see that $\alpha \sim \alpha_{\mathrm{pp}}$. Thus α_{pp} is a purely periodic (hence the "pp") irrational equivalent to α. Applying Galois's theorem we find that

$$-1/\widetilde{\alpha_{\mathrm{pp}}} = \left[\,\overline{\left(\overleftarrow{W_2}, \overleftarrow{W_1}\right)}\,\right],$$

which in view of the palindromic nature of W_1 and W_2 implies that

$$-1/\widetilde{\alpha_{\mathrm{pp}}} = \left[W_2, \overline{(W_1, W_2)}\right].$$

Exposed in this light, it becomes immediately clear that $-1/\widetilde{\alpha_{\mathrm{pp}}} \sim \alpha_{\mathrm{pp}} \sim \alpha$. Of course by displaying $-1/\widetilde{\alpha_{\mathrm{pp}}}$ in the ridiculously ornate guise

$$-\frac{1}{\widetilde{\alpha_{\mathrm{pp}}}} = \frac{-1 + 0\,\widetilde{\alpha_{\mathrm{pp}}}}{0 + 1\,\widetilde{\alpha_{\mathrm{pp}}}}$$

and recalling (3), we recognize that $-1/\widetilde{\alpha_{\mathrm{pp}}} \sim \widetilde{\alpha_{\mathrm{pp}}}$. We infer that $\alpha \sim \widetilde{\alpha_{\mathrm{pp}}}$.

Suppose next that μ and ν are quadratic irrationals related by

$$\mu = \frac{A + B\nu}{C + D\nu}$$

for integers A, B, C, and D. Then by conjugating both sides of this relation (in the sense of "taking algebraic conjugates"), we find that

$$\widetilde{\mu} = \widetilde{\left(\frac{A + B\nu}{C + D\nu}\right)} = \frac{\widetilde{(A + B\nu)}}{\widetilde{(C + D\nu)}} = \frac{\widetilde{A} + \widetilde{B\nu}}{\widetilde{C} + \widetilde{D\nu}} = \frac{A + B\widetilde{\nu}}{C + D\widetilde{\nu}}.$$

(The verification that conjugation is so compatible with ordinary arithmetical operations is a whole other tale that we will not recount here.) The upshot of this calculation: for quadratic irrationals μ and ν, $\mu \sim \nu$ if and only if $\widetilde{\mu} \sim \widetilde{\nu}$. In particular, since we know that $\alpha_{\mathrm{pp}} \sim \alpha$, we now see that $\widetilde{\alpha_{\mathrm{pp}}} \sim \widetilde{\alpha}$. Hence, as $\alpha \sim \widetilde{\alpha_{\mathrm{pp}}}$, we realize that $\alpha \sim \widetilde{\alpha_{\mathrm{pp}}} \sim \widetilde{\alpha}$, which tells half the story.

Finally, we assume that $\alpha \sim \widetilde{\alpha}$. By Lagrange's theorem we are able to write

$$\alpha = \left[a_0, a_1, \dots, a_M, \overline{W}\,\right]$$

for some word $W = (w_1, w_2, \dots, w_L)$. If we set

$$\alpha_{\mathrm{pp}} = \left[\,\overline{W}\,\right] = [w_1, w_2, \dots, w_{L-1}, w_L, w_1, w_2, \dots, w_{L-1}, w_L, \dots],$$

then Galois's theorem ensures that

$$-1/\widetilde{\alpha_{\mathrm{pp}}} = \left[\,\overline{\overleftarrow{W}}\,\right] = [w_L, w_{L-1}, \dots, w_2, w_1, w_L, w_{L-1}, \dots, w_2, w_1, \dots].$$

Given our supposition that $\alpha \sim \widetilde{\alpha}$, we also deduce that

$$\alpha_{\mathrm{pp}} \sim \alpha \sim \widetilde{\alpha} \sim \widetilde{\alpha_{\mathrm{pp}}} \sim -1/\widetilde{\alpha_{\mathrm{pp}}}.$$

Thus, by appealing to Serret's theorem, we conclude that from some point onward the continued fraction expansion of α_{pp} agrees with that of $-1/\widetilde{\alpha_{\mathrm{pp}}}$. That is, there exists some index K ($1 \leq K \leq L$) such that

$$\begin{aligned} w_1 &= w_K \\ w_2 &= w_{K-1} \\ w_3 &= w_{K-2} \\ &\vdots \\ w_{K-1} &= w_2 \\ w_K &= w_1 \\ \hline w_{K+1} &= w_L \\ w_{K+2} &= w_{L-1} \\ &\vdots \\ w_{L-1} &= w_{K+2} \\ w_L &= w_{K+1}. \end{aligned}$$

From this string of identities we learn that W can be partitioned into the two subwords

$$(w_1, w_2, \ldots, w_{K-1}, w_K)$$

and

$$(w_{K+1}, w_{K+2}, \ldots, w_{L-1}, w_L),$$

each of which is a palindrome. It follows that α does indeed have a tail of two palindromes, thus closing the book on our proof. ■

5. A TALE OF TWO CHALLENGES $\left[\ldots, \overline{\text{Yo, banana boy!; Too hot to hoot?}}\right]$. If two quadratic irrationals α and β are equivalent, then Serret's theorem informs us that, after some initial run of terms, the continued fraction expansions of α and β coincide. We now note that if $\alpha \sim \beta$, then the integers A, B, C, and D satisfying the conditions of (3) spring organically from the initial runs of those dissagreeable terms in the respective continued fraction expansions by cultivating the associated matrices of the form (6). The details of this organic procedure provide a natural sequel to our story that readers might be interested in pursuing on their own.

We end with the odd twist that the special tail of two palindromes (W, b) appearing in Theorem 1, in which the second palindrome consists of a single letter, is actually not all that special. In fact, we assert the following:

Theorem 3. *If α has a tail of two palindromes and at least one of the two palindromic words is of odd length, then there exists a palindrome W and an integer b such that $\alpha \sim \left[\overline{(W, b)}\right]$.*

The proof of this result is left as the concluding cliff-hanger to our tale.

As we turn the final page on our story, we detect a faint voice riding the gentle breeze—the voice of a quadratic irrational from some far-off field who has tirelessly bent over backwards and forwards in the hopes of finding some close connection with its long-lost conjugate. We hear that voice at last proclaim:

> *It is a far, far better thing that I do, than I have ever done;*
> *it is a far, far better rest that I go to than I have ever known.*

The End.

DEDICATION. This paper is dedicated to the memory of Johnny Carson, who lifted our lives and made us laugh.

REFERENCES

1. E. B. Burger, *Exploring the Number Jungle: A Journey into Diophantine Analysis*, American Mathematical Society, Providence, 2000.
2. E. Galois, Démonstration d'un théorème sur les fractions continues périodiques, *Annales de mathématiques* (1829) 1–8.
3. J. L. Lagrange, Additions au mémoire sur la résolution des équations numériques, *Mém. Berl.* **24** (1770) 603–615.
4. J.-A. Serret, Sur le dévelopement en fraction continue de la racine carrée d'un nombre entier, *J. Math. Pures Appl.* **12** (1847) 518–520.

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