
A Pascal-Type Triangle Characterizing Twin Primes

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1. INTRODUCTION. Two of the most ubiquitous objects in mathematics are the sequence of prime numbers and the binomial coefficients (and thus Pascal's triangle). A connection between the two is given by a well-known characterization of the prime numbers:

Consider the entries in the k th row of Pascal's triangle, without the initial and final entries. They are all divisible by k if and only if k is a prime.

It is the purpose of this article to present a triangular array of numbers similar to Pascal's triangle and to prove a corresponding criterion for the *twin prime pairs*. A further goal is to place all this in the context of some classical orthogonal polynomials and to relate it to some recent work of John D'Angelo.

To begin, and for the sake of completeness, we present a short proof of the Pascal triangle criterion. First suppose that $k = p$ is prime. Then we see that in

$$\binom{p}{j} = \frac{p!}{j!(p-j)!} \quad (1 \leq j \leq p-1)$$

the numerator is divisible by p , while the denominator is not. On the other hand, suppose that k is not prime. Choose a prime divisor p of k and write $k = mp^\alpha$, where $p \nmid m$. Then we see that

$$\binom{k}{p} = \frac{mp^\alpha(mp^\alpha - 1) \cdots (mp^\alpha - p + 1)}{1 \cdot 2 \cdots p} \quad (1)$$

is not divisible by p^α , and therefore not by k .

Other related primality criteria are known. A particularly intriguing one can be found in [14] and [10].

In constructing a twin-prime analogue of the Pascal triangle criterion we shall emulate the two parts of the foregoing proof. What makes the second part work is the choice of a prime divisor p of the composite number k . In analogy, if the integers $2k - 1$ and $2k + 1$ are not a twin-prime pair, we can find a prime p properly dividing one of these two integers. We now need to find an expression depending on k and analogous to (1) that is not divisible by p . This is achieved by the following lemma:

Lemma 1. *Let p be an odd prime properly dividing either $2k - 1$ or $2k + 1$. Then p does not divide the binomial coefficient*

$$\binom{k + (p-3)/2}{p-2}.$$

We prove this with an argument similar to the previous one. If $p \mid 2k - 1$, we can write $2k - 1 = p(2m + 1)$ with $m \geq 1$. Then $k + (p - 3)/2 = p - 1 + mp$, and the binomial coefficient becomes

$$\frac{(p - 1 + mp)(p - 2 + mp) \cdots (2 + mp)}{(p - 2)!}.$$

In case $p \mid 2k + 1$, the same procedure yields

$$\frac{(p - 2 + mp)(p - 3 + mp) \cdots (1 + mp)}{(p - 2)!}.$$

In either event the numerator is clearly relatively prime to p .

Now, in order to have the desired criterion, we employ the binomial coefficient

$$\binom{k + s}{2s + 1}.$$

With $s = (p - 3)/2$ this is the expression in Lemma 1. To make the other direction work, we still need to modify this binomial coefficient. In fact, our main object of study will be the expressions

$$a(k, s) := \binom{k + s}{2s + 1} \frac{(2k - 1)(2k + 1)}{2s + 3}, \tag{2}$$

where $k \geq 1$ and $0 \leq s \leq k - 1$. The numerator of the additional factor is there to create divisibility, whereas the denominator is there to inhibit divisibility. These numbers $a(k, s)$ form a triangular array, where k corresponds to the row number:

			1			
		10	3			
	35	28	5			
	84	126	54	7		
165	396	297	88	9		
286	1001	1144	572	130	11	
455	2184	3510	2600	975	180	13
.

(3)

We first observe that all the entries shown are integers. This is not clear from the definition (2), in the same way that it is not obvious that the Catalan numbers

$$\frac{1}{n + 1} \binom{2n}{n}$$

are integers. However, in the next section we establish the following fact:

Theorem 1. *All the entries in the triangle (3) are integers (i.e., $a(k, s)$ belongs to \mathbb{N} whenever $k \geq 1$ and $0 \leq s \leq k - 1$).*

Now consider, for example, row six in the triangle. We observe that all entries but one (the second from the right) are divisible by $2k - 1 = 11$, and similarly, all but one (the right-most entry) are divisible by $2k + 1 = 13$. On the other hand, this property does not hold for row five (the factor 11 has the divisibility property, but 9 does not).

This is explained by the following theorem, which is our main result:

Theorem 2. *The entries in the k th row of the number triangle (3) are divisible by $2k - 1$ with exactly one exception and are divisible by $2k + 1$ with exactly one exception if and only if $(2k - 1, 2k + 1)$ is a pair of twin primes.*

Thus the number triangle (3) characterizes the twin primes in a way similar to that in which Pascal's triangle characterizes the primes. It should be noted that Matiyasevich [16] derived criteria for primality and for twin primes that also involved binomial coefficients. However, these criteria are of quite a different nature.

To prove Theorem 2 we use Lemma 1 and the fact that, by Theorem 1, the numbers $a(k, s)$ are all integers. The last two terms in the k th row of (3) are

$$a(k, k - 2) = (2k - 2)(2k + 1), \quad a(k, k - 1) = 2k - 1.$$

If $(2k - 1, 2k + 1)$ is a twin prime pair, it is clear that $a(k, k - 1)$ is not divisible by $2k + 1$ and that $a(k, k - 2)$ is not divisible by $2k - 1$. All other $a(k, s)$ are divisible by both since $2s + 3 < 2k - 1$.

On the other hand, if $2k - 1$ is not a prime, it has a proper odd prime factor of the form $2s + 3$ with $0 \leq s \leq k - 3$ (in fact, $s < (k - 3)/2$), and this factor is relatively prime to $2k + 1$. As already indicated after Lemma 1, we set $p = 2s + 3$, so that

$$\binom{k + s}{2s + 1} = \binom{k + \frac{p-3}{2}}{p - 2}.$$

By Lemma 1, p does not divide this binomial coefficient; hence

$$a\left(k, \frac{p - 3}{2}\right)$$

is not divisible by $2k - 1$.

Similarly, if $2k + 1$ is not prime, it, too, has a proper odd prime factor of the form $2s + 3$ with $0 \leq s \leq k - 3$ (in this instance, $s \leq (k - 3)/2$). This factor is relatively prime to $2k - 1$. We write $p = 2s + 3$ and proceed as earlier to reach the same conclusion. This completes the proof.

2. A GENERATING FUNCTION. Counting arguments are often used to prove that a combinatorial expression is in fact an integer. Probably the best known argument of this type consists in showing that $\binom{n}{k}$ counts the number of ways to choose a k -element subset from an n -element set. Similarly, the Catalan numbers

$$\frac{1}{n + 1} \binom{2n}{n}$$

can be shown to be integers through any of their numerous combinatorial interpretations (see, for example, [11, p. 358ff.]). A good introduction to this kind of combinatorial reasoning is given in the recent book [3]. Demonstrating that $a(k, s)$ counts something would obviously prove Theorem 1.

Here we use another important and more general method, namely, generating functions. Basically we consider the entries in a row of our number triangles as coefficients of polynomials and then study these polynomial sequences. We illustrate this with the

Pascal triangle. Using a geometric series, we have

$$\frac{1}{1 - (1 + x)t} = \sum_{k=0}^{\infty} P_k(x)t^k = 1 + (1 + x)t + (1 + 2x + x^2)t^2 + \dots$$

We can thus rephrase the Pascal triangle criterion as follows:

When $k \geq 2$ the coefficients of $P_k(x)$ are divisible by k with exactly two exceptions if and only if k is prime.

As an aside we mention that $P_k(x) = (1 + x)^k$ hints at another equivalent formulation of the original Pascal triangle primality criterion: the integer p is prime if and only if the polynomial congruence

$$(x + 1)^p \equiv x^p + 1 \pmod{p}$$

holds. This congruence is important in algebra and number theory (see, for example, [12, p. 68]). It has recently gained additional prominence in connection with the AKS primality test, a remarkable new algorithm for testing primality of an integer “in polynomial time” (see [2] or [4]).

In analogy with the polynomials $P_k(x)$, we define polynomials $Q_k(x)$ with the entries of the k th row of the triangle (3) as coefficients:

$$Q_k(x) := \sum_{s=0}^{k-1} a(k, s)x^s = \sum_{s=0}^{k-1} \binom{k+s}{2s+1} \frac{(2k-1)(2k+1)}{2s+3} x^s. \quad (4)$$

Thus $Q_1(x) = 1$, $Q_2(x) = 10 + 3x$, $Q_3(x) = 35 + 28x + 5x^2$, $Q_4(x) = 84 + 126x + 54x^2 + 7x^3$, etc.

We now find the generating function for the $Q_k(x)$ that is the analogue of the generating function for the $P_k(x)$.

Lemma 2. *The polynomial sequence $Q_k(x)$ satisfies*

$$t \frac{1 + (6 + x)t + t^2}{(1 - (2 + x)t + t^2)^2} = \sum_{k=1}^{\infty} Q_k(x)t^k. \quad (5)$$

The shape of the generating function (i.e., of the left-hand side of (5)) indicates a connection with certain orthogonal polynomials. This could be used to prove Lemma 2, and we shall return to this connection later. However, here we present a more straightforward and standard proof.

Multiplying both sides of (5) by the denominator of the left-hand side, namely,

$$1 - (4 + 2x)t + (6 + 4x + x^2)t^2 - (4 + 2x)t^3 + t^4,$$

we see that it defines a sequence of polynomials $Q_k(x)$ satisfying the following equations:

$$\begin{aligned} Q_1(x) &= 1, \\ Q_2(x) - (4 + 2x)Q_1(x) &= 6 + x, \\ Q_3(x) - (4 + 2x)Q_2(x) + (6 + 4x + x^2)Q_1(x) &= 1, \\ Q_4(x) - (4 + 2x)Q_3(x) + (6 + 4x + x^2)Q_2(x) - (4 + 2x)Q_1(x) &= 0, \end{aligned}$$

and, in general,

$$Q_{k+4}(x) - (4 + 2x)Q_{k+3}(x) + (6 + 4x + x^2)Q_{k+2}(x) - (4 + 2x)Q_{k+1}(x) + Q_k(x) = 0. \tag{6}$$

Moreover, these equations recursively define a unique sequence of polynomials. Hence we simply need to show that the expression on the right of (4) satisfies these equations. Replace each occurrence of Q_k with that expression and collect equal powers of x . This yields a straightforward (though tedious) verification and can be done either “by hand” or with the help of a computer algebra system.

This method also gives an immediate proof by induction of Theorem 1, to wit, that the coefficients of the $Q_k(x)$ are integers. Indeed, we have seen that this is true for $k = 1, 2, 3$, and 4. The induction step is now provided by identity (6).

3. A ONE-PRIME VARIANT AND ORTHOGONAL POLYNOMIALS. While studying resultants and discriminants of certain linear combinations of Chebyshev polynomials (see [8]), the second author found that his colleague John D’Angelo was studying some of the same polynomials in connection with problems in the theory of several complex variables. In the ensuing conversations, D’Angelo pointed out to the present authors the following result and its proof:

Theorem 3. *The number triangle whose k th row consists of the numbers*

$$b(k, s) = \binom{k+s}{2s} \frac{2k+1}{2s+1} \quad (0 \leq s \leq k) \tag{7}$$

characterizes the primes in the following sense: the entries in the k th row are divisible by $2k + 1$ with exactly one exception if and only if $2k + 1$ is prime.

This theorem played a pivotal role in our making the transition from Pascal’s triangle to the triangle (3). As before we place the numbers $b(k, s)$ in a triangle beginning with

$$\begin{array}{cccccccc}
 & & & & & & & 1 \\
 & & & & & & & & 3 & & 1 \\
 & & & & & & & & 5 & & 5 & & 1 \\
 & & & & & & & & 7 & & 14 & & 7 & & 1 \\
 & & & & & & & & 9 & & 30 & & 27 & & 9 & & 1 \\
 & & & & & & & & 11 & & 55 & & 77 & & 44 & & 11 & & 1 \\
 & & & & & & & & 13 & & 91 & & 182 & & 156 & & 65 & & 13 & & 1 \\
 & & & & & & & & \dots & & \dots & & \dots & & \dots & & \dots & & \dots & & \dots
 \end{array} \tag{8}$$

This triangle, in slightly modified form, can also be found in [5, p. 175]. There it is shown how the triangle’s entries arise in the proof of the existence of a “proper polynomial mapping” (for definitions, see [5, p. 151]) between the unit balls in \mathbb{C}^2 and in \mathbb{C}^{2+k} .

Once again we define a sequence of polynomials $J_k(x)$ that have the entries in the k th row as coefficients. (Note that here the rows are counted starting with $k = 0$.) Thus

$$J_k(x) = \sum_{s=0}^k \binom{k+s}{2s} \frac{2k+1}{2s+1} x^s.$$

While it is not obvious that the coefficients of these polynomials are integers, this fact can be verified by induction, just as in the proof of Theorem 1. For this purpose we use the generating function

$$\frac{1+t}{1-(2+x)t+t^2} = \sum_{k=0}^{\infty} J_k(x)t^k, \tag{9}$$

which can be derived in a way similar to (but easier than) the derivation in Lemma 2. The entries in the triangle (8) also have a combinatorial interpretation [18].

The polynomials $J_k(x)$ are not new. They have an important extremal property relating the degrees of certain polynomials to the number of their terms (see [7]).

Theorem 3 can be proved in the same way as Theorem 2—the proof is in fact simpler. The analogue of Lemma 1 is the fact that an odd prime p that properly divides $2k+1$ cannot divide the binomial coefficient

$$\binom{k+(p-1)/2}{p-1}.$$

To see this we write $2k+1 = p(2m+1)$ and observe that

$$\binom{k+\frac{p-1}{2}}{p-1} = \frac{(mp+p-1)(mp+p-2)\cdots(mp+1)}{(p-1)!}.$$

The remainder of the proof is similar to that of Theorem 2.

The polynomials $J_k(x)$ are also closely related to the *Chebyshev polynomials of the second kind* $U_k(x)$, where the first few such are $U_0(x) = 1$, $U_1(x) = 2x$, and $U_2(x) = 4x^2 - 1$. These polynomials occur, for instance, in the expansion of $\sin[(k+1)\theta]/\sin\theta$ in terms of powers of $\cos\theta$, thus generalizing the double-angle formula $\sin(2\theta) = 2\sin\theta\cos\theta$. They also belong to the class of classical orthogonal polynomials and have numerous interesting properties and important applications in approximation theory and numerical analysis (see, for instance, [17]). Their generating function is remarkably simple:

$$\frac{1}{1-2xt+t^2} = \sum_{k=0}^{\infty} U_k(x)t^k.$$

By comparing this with (9) and using standard identities for the Chebyshev polynomials (see [17] or [1, chap. 22]) one obtains

$$J_k(x) = U_{2k}\left(\frac{\sqrt{4+x}}{2}\right).$$

These polynomials may also be thought of from the point of view of circulants. For this we refer readers to [13].

Next, the polynomials $Q_k(x)$ of (4) and (5) are related to the *Gegenbauer* (or *ultraspherical*) *polynomials*, a class of polynomials that generalize the Chebyshev polynomials $U_k(x)$ in a natural way. They can be defined by the generating function

$$\frac{1}{(1-2xt+t^2)^\alpha} = \sum_{k=0}^{\infty} C_k^{(\alpha)}(x)t^k.$$

For fixed nonzero real parameters α greater than $1/2$ they also belong to the classical orthogonal polynomials and have numerous remarkable properties and applications (see [1, chap. 22]). By expanding the denominator of (5) according to the aforementioned formula with $\alpha = 2$ we obtain

$$Q_k(x) = C_{k-1}^{(2)}\left(\frac{2+x}{2}\right) + (6+x)C_{k-2}^{(2)}\left(\frac{2+x}{2}\right) + C_{k-3}^{(2)}\left(\frac{2+x}{2}\right).$$

4. TRIANGULAR RECURRENCE RELATIONS. We began this paper by considering the Pascal triangle, which can be constructed from the important relation

$$\binom{k+1}{s} = \binom{k}{s-1} + \binom{k}{s}.$$

The question then arises whether the number triangles (3) and (8) are associated with similar “triangular” recurrence relations. Such relations do indeed exist: the numbers $a(k, s)$ defined in (2) satisfy

$$a(k+1, s) = \frac{k+s+1}{(2k-1)(2k)} [(k+s)a(k, s-1) + 2(2k+2s+3)a(k, s)], \quad (10)$$

and the numbers $b(k, s)$ that are encountered in (7) satisfy

$$b(k+1, s) = \frac{2(k+s)(k+s+1)}{(2k-1)(2k)(2k+1)(2k+2)} \times [(2k^2+2k+3-3s)b(k, s-1) + 2(4k^2+4k-3-6s)b(k, s)]. \quad (11)$$

Along with the initial values $a(1, 0) = b(0, 0) = 1$ and the usual convention that all terms “off the triangle” are zero, these relations could be used to construct the triangles (3) and (8). Of course, these do look rather convoluted compared with the triangular recurrence for the Pascal triangle.

Given the denominators on the right-hand sides of (10) and (11), it is by no means clear that the numbers $a(k, s)$ and $b(k, s)$, if they were defined in this way, should all be integers. However, we did already establish that this is the case. There has recently been interest in other recurrences that produce integers for no obvious reasons (see, for example, [9]).

Both relations (10) and (11) are easy to verify by substituting into them the expressions (2) and (7), respectively, then simplifying and using standard identities for the binomial coefficients. These relations were discovered by setting up the equation (in the case of (11))

$$b(k+1, s) = A \cdot b(k, s-1) + B \cdot b(k, s)$$

with undetermined coefficients A and B . After canceling the various factorials common to $b(k+1, s)$, $b(k, s-1)$, and $b(k, s)$, we used a computer-algebra system to find A and B with the aid of the extended Euclidean algorithm. Relation (10) was also first obtained in this way.

While the relations (10) and (11) are very complicated, much simpler relations for the $a(k, s)$ can be obtained from (6) by equating coefficients in (4). Similarly, simple relations for the $b(k, s)$ can be obtained from (9) by the same method. We leave the details to the interested reader. These formulas differ from (10) and (11) in that they

involve more than just two adjacent rows in the corresponding triangles. Another recurrence relation for the $b(k, s)$, involving all the preceding rows in the triangle, can be found on page 175 of [5].

In this connection it is interesting to note that triangular recurrence relations, and more generally “recurrence double sequences,” have been studied in great generality and from the points of view of number theory, combinatorics, and computer science. For some recent results on this topic, see [15] and the references listed there.

5. OTHER PRIME PATTERNS. It is a natural question to ask whether there are triangles analogous to (3) that characterize prime pairs with fixed differences larger than two. Similarly, one could ask whether prime k -tuples with fixed configurations—for instance, prime triples of the form $(2k + 1, 2k + 3, 2k + 7)$ —can be characterized in this way.

As far as the first question is concerned, it is not difficult to derive an analogue to Lemma 1, but we were unable to find an analogue to (2) with the required properties. For the second question, and with the properties of the generating function in mind (that is, (9) for single primes and (5) for twin primes), one may wish to study number triangles generated by analogues of (9) and (5) with higher powers in the denominator. We did indeed study such triangles from a different perspective but did not detect any characterizations of prime k -tuples, nor were we able to adapt our method to achieve such characterizations. It therefore remains an interesting open problem whether there is an analogue of triangle (3) for prime k -tuples other than the twin primes.

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Public Announcement

The Federal Government recommends that numbers greater than 2^{40} have their primality tested. The odd ones need the PSW test, and all need an MR test at regular intervals. Those with a history of pseudoprimality should also have the AKS test. These tests are now quick and painless. Call your local number theorist to schedule an appointment.

—Submitted by Jane Ross and Felipe Voloch,
University of Texas at Austin