
Touring the Calculus Gallery

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1. INTRODUCTION. The premise of this article—what our friends in theatre would call its “conceit”—is that we are about to visit a gallery devoted to the history of calculus. Admittedly no such institution exists, but it is easily imagined. One need only think of an art museum, albeit one whose masterpieces are not canvases but theorems and whose masters are not Courbet and Cezanne but Leibniz and Lebesgue.

Our stroll through the Calculus Gallery must be brief, and we can stop only occasionally to examine particular works in detail. Even so, the visit should provide a glimpse of the development of calculus/analysis from its appearance in the late seventeenth century, through its expansion in the eighteenth, to its “Classical Period” in the first two-thirds of the nineteenth, and on to the mature subject of today.

Like any museum, we enter through a grand hall whose walls are inscribed with noble sentiments. One, from John von Neumann, catches our eye [16, p. 3]:

The calculus was the first achievement of modern mathematics, and it is difficult to overestimate its importance.

In response to such an encomium, we have but one choice: pay the admission and take a look around.

2. THE NEWTON ROOM. Calculus has many antecedents. Some reach back to classical times with the work of Eudoxus or Archimedes, and certainly seventeenth-century Europe saw important contributions from the likes of Fermat, Pascal, and Barrow. But it was Isaac Newton (1642–1727) who first cobbled the assorted ideas into a unified subject, and so it is in the gallery’s Newton Room that we start.

Some dates may be helpful. Newton’s discoveries began in the mid-1660s while he was a student at Trinity College, Cambridge. His first attempt to put these thoughts on paper resulted in the “October, 1666 tract,” which he subsequently refined and expanded as the *De analysi* of 1669 and further polished into the *Methodus fluxionum et serierum infinitarum* of 1671. These writings treated maxima and minima, infinite series, tangents, and areas—still the basic topics of elementary calculus.

The excerpt shown in Figure 1 is taken from the *De analysi* [22, p. 20]. Here Newton considered the arc of a circle of radius 1 centered at the origin. Denoting the length of circular arch αD by z , he sought “the Base from the Length of the Curve,” that is, an expression for the length of segment AB in terms of z . Because z is the arclength on the unit circle, it is also the (radian) measure of $\angle \alpha AD$. Thus $x = \overline{AB} = \sin z$.

Through a series of manipulations too convoluted to include here, he showed that

$$\sin z = x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \frac{1}{362880}z^9 - \dots,$$

and for good measure also provided the cosine series, easily visible in section 46 of the excerpt:

$$\cos z = A\beta = \sqrt{1 - x^2} = 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 + \frac{1}{403200}z^8 - \frac{1}{3628800}z^{10} + \dots.$$



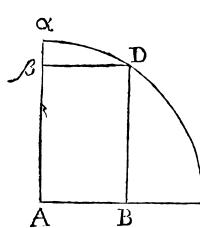
Newton

Mathematics historian Derek Whiteside observed that these famous series “here appear for the first time in a European manuscript” [23, p. 237]. (His qualification recognized the discovery of equivalent formulas by Indian mathematicians more than a century before, albeit in documents unknown to Westerners during Newton’s time. See [4].)

It is reasonable to presume that Newton’s triumphs swept the world in 1669. Of course, it did not happen that way. In spite of their mathematical brilliance, his results remained unpublished for decades. For reasons as much psychological as scientific, Newton chose not to share these discoveries in print, and so the glory of first publication would fall to another.

Before continuing this tale, we should say a word about Newton’s logical justification of his methods. In particular, we address his description of the “ultimate ratio of vanishing quantities”—what we now call the derivative. Perceiving this as a quotient

To find the Base from the Length of the Curve given.



45. If from the Arch αD given the Sine AB was required; I extract the Root of the Equation found above, viz. $z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7$ (it being supposed that $AB = x$, $\alpha D = z$, and $A\alpha = 1$) by which I find $x = z - \frac{1}{6}z^3 + \frac{1}{12}z^5 - \frac{1}{504}z^7 + \frac{1}{3024}z^9$ &c.

46. And moreover if the Cofine $A\beta$ were required from that Arch given, make $A\beta (= \sqrt{1 - xx}) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8 - \frac{1}{362880}x^{10}$, &c.

Figure 1. Newton’s series for sine and cosine (1669).

whose numerator and denominator were shrinking to zero, Newton characterized the *ultimate* ratio as “the ratio of the quantities not before they vanish, nor afterwards, but with which they vanish” [17, p. 300].

This is highly problematic. We might agree that the derivative is not the ratio of quantities *before* they vanish, but what are we to make of the ratio *after* they have vanished? One is tempted to ask, “How long after?” A minute? A week? It seems that Newton wanted to capture the ratio at that precise instant when—poof!—numerator and denominator simultaneously disappear into nothingness. His ideas, although conveying a useful imagery, would surely have to be revisited.

But such criticism is insignificant compared to Newton’s towering achievements. He later described his days of youthful discovery as those when he was “in the prime of my age for invention and minded Mathematicks and Philosophy more than at any time since” [21, p. 143]. For that period of invention, mathematicians will be forever grateful.

3. THE LEIBNIZ ROOM. As observed, it was not Newton who taught the world calculus. That distinction belongs to his celebrated contemporary, Gottfried Wilhelm Leibniz (1646–1716). The multitalented Leibniz had reached adulthood with only modest mathematical training, a situation he recalled in these words [7, p. 11]:

When I arrived in Paris in the year 1672, I was self-taught as regards geometry, and indeed had little knowledge of the subject, for which I had not the patience to read through the long series of proofs.



Leibniz

With guidance from Christian Huygens (1629–1695), Leibniz sought to remedy this defect. Those daunting proofs notwithstanding, he wrote that

[I]t seemed to me, I do not know by what rash confidence in my own ability, that I might become the equal of these if I so desired [7, p. 12].

Because few subjects could withstand his blinding intellect, Leibniz moved from novice to master in short order. Within a few years, he had created the calculus.

Figure 2 is a diagram that Leibniz used in an argument from 1674 [7, p. 42]. It is included here for two reasons. First, with all its curves and triangles, it illustrates the geometrical flavor of the subject in those formative days. Second, although its purpose may be far from evident, it lay behind one of Leibniz's great early discoveries, for from this tangle of lines he deduced that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots = \frac{\pi}{4}$.

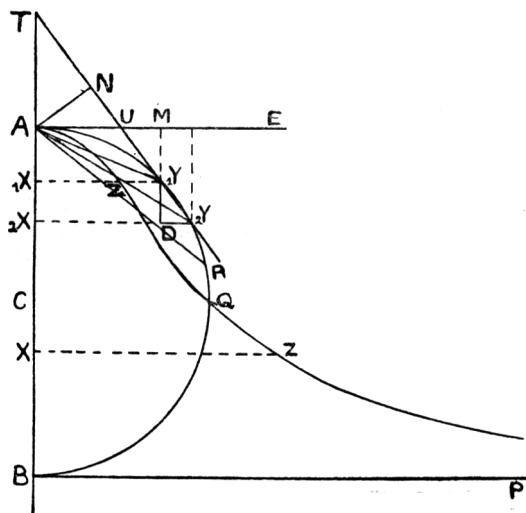


Figure 2. Leibniz excerpt (1674).

This result, the so-called Leibniz series, is worthy of a place of honor in our museum. The left side of the equation displays a trivial pattern; the right side, half of half of π , is anything but obvious. It was believed that here “for the first time . . . the area of a circle was exactly equal to a series of rational quantities” [7, p. 47]. We might take issue with this use of “exactly,” but it is hard not to echo the reaction of Huygens who, in [7, p. 46],

praised [the proof] very highly, and when he returned the dissertation said, in the letter that accompanied it, that it would be a discovery always to be remembered among mathematicians. . . .

Probably no artifact in the Calculus Gallery holds a more venerated place than the document shown in Figure 3 [17, p. 273]. This was the first page of the first publication on calculus. In the October 1684 issue of the *Acta eruditorum*, Leibniz presented a “new method” for finding maxima, minima, and tangents and, in the last line of the title, promised “a remarkable type of calculus for this.” The name stuck.

His article was not easy reading. A look at the second paragraph reveals Leibniz’s presentation of such rules as:

- $d(ax) = a dx$
- $d(z - y + vv + x) = dz - dy + d(vv) + dx$
- $d(xv) = x dv + v dx$

Sprinting through formulas at breakneck speed, he introduced a calculus that may not have been lively but most certainly was lean.

MENSIS OCTOBRIS A. M DCLXXXIV. 467
 NOVA METHODVS PRO MAXIMIS ET MINIMIS,
 itemque tangentibus, quae nec fractas, nec irrationales quantitates moratur, & singulare pro illis calculi genus, per G.G. L.

Sit axis AX, & curvæ plures, ut VV, WW, YY, ZZ, quarum ordinatae, ad axem normales, VX, WX, YX, ZX, quæ vocentur respectivæ, v, vv, y, z; & ipsa AX abscissa ab axe, vocetur x. Tangentes sunt VB, WC, YD, ZE axi occurrentes respectivæ in punctis B, C, D, E. Jam recta aliqua pro arbitrio assumta vocetur dx , & recta quæ sit ad dx , ut v (vel vv , vel y , vel z) est ad VB (vel WC, vel YD, vel ZE) vocetur $d v$ (vel $d vv$, vel dy vel dz) sive differentia ipsarum v (vel ipsarum vv , aut y , aut z). His positis calculi regulæ erunt tales:

TAB. XII.

Sit a quantitas data constans, erit da æqualis o, & d ax erit æquæ $\frac{d}{d x}$: si sit y æquæ v (seu ordinata quævis curvæ YY, æqualis cuivis ordinatae respondentis curvæ VV) erit dy æquæ dv . Jam *Additio* & *Subtractione*: si sit $z = y + vv$ & x æquæ v , erit $dz = dy + vv + dx$, seu dv , æquæ $dz - dy = dv + dx$. *Multiplicatio*, $dx \cdot v$ æquæ $x \cdot dv + v \cdot dx$, seu positio y æquæ xv , fiet dy æquæ $x \cdot dv + v \cdot dx$. In arbitrio enim est vel formulam, ut xv , vel compendio pro ea literam, ut y , adhibere. Notandum & x & dx eodem modo in hoc calculo tractari, ut y & dy , vel aliam literam indeterminatam cum sua differentiali. Notandum etiam non dari semper regressum a differentiali *Æquatione*, nisi cum quadam cautio-

v v
 $\frac{d}{d y} + y \frac{d}{d v}$

$\frac{d}{d v} + v \frac{d}{d x}$

Figure 3. Leibniz's first paper on differential calculus (1684).

4. THE L'HOSPITAL EXHIBIT. By 1684 the calculus had been (twice) discovered and had been described in a journal. The next step was a textbook, a way of organizing and clarifying Leibniz's dense ideas. The first text appeared in 1696 under the title *Analyse des infinitement petits pour l'intelligence des lignes courbes* (Analysis of the infinitely small for the understanding of curved lines). Its author was the Marquis de l'Hospital (1661–1704). Not a mathematician of the highest rank, l'Hospital acquired much of his material through a financial arrangement with Johann Bernoulli (1667–1748) in which the latter, for a fee, provided him with lectures that were to become this book. It is important to note that l'Hospital was candid about the source of his work. Referring to Leibniz and the Bernoullis, he wrote [17, p. 312]:

I have made free use of their discoveries so that I frankly return to them whatever they please to claim as their own.

He began with some definitions, chief among which was that of the differential. According to l'Hospital [15, p. 2],

The infinitely small part by which a variable quantity is continually increased or decreased is called the differential (*Difference*) of that quantity.

In modern notation, if $y = y(x)$ and if x is increased by its infinitely small differential dx , then the corresponding, infinitely small change in y is $dy = y(x + dx) - y(x)$

and so

$$y(x + dx) = y + dy. \quad (1)$$

The notion of infinite smallness does not appeal to modern sensibilities any more than does Newton's idea of vanishing quantities. But l'Hospital used it to derive key results of differential calculus, among them the product rule.

Theorem. *The differential of xy is $y dx + x dy$.*

Proof. As x becomes $x + dx$, we see that y becomes $y + dy$. It follows that the product xy becomes $(x + dx)(y + dy)$, and so the differential of xy is

$$\begin{aligned} d(xy) &= (x + dx)(y + dy) - xy = [xy + y dx + x dy + dx dy] - xy \\ &= y dx + x dy + dx dy. \end{aligned}$$

L'Hospital then jettisoned the $dx dy$ term because it "is an infinitely small quantity with respect to both terms $x dy$ and $y dx$." He was left with the fact that "the differential of the product of two quantities is equal to the product of the differential of the first by the second quantity plus the product of the differential of the second by the first" [15, p. 4]. ■

Skeptical readers may wish to debate what exactly happened to that $dx dy$, but l'Hospital was in no mood to quibble. He moved quickly to his next topic, the quotient rule.

Corollary. *The differential of x/y is*

$$\frac{y dx - x dy}{yy}.$$

(In those days, the square of a variable was written as the product of the variable by itself.)

Proof. Introduce $z = x/y$, so that $x = yz$. By the product rule, we know that $dx = y dz + z dy$ and thus

$$d\left(\frac{x}{y}\right) = dz = \frac{dx - z dy}{y} = \frac{dx - (x/y) dy}{y} = \frac{y dx - x dy}{yy}. \quad \blacksquare$$

From there, his text ranged across topics now standard in differential calculus. Section 2 was devoted to tangent lines, section 3 to maxima and minima, and section 4 to inflection points. Some of his problems are perfectly suitable for a modern course, such as, "Among all cones that can be inscribed in a sphere, find that which has the greatest convex surface" [15, pp. 45–46]. In his solution, l'Hospital set the appropriate differential equal to zero and found that the height of the largest cone is two-thirds the diameter of the sphere. This 300-year-old max/min problem has lost none of its luster.

Before moving on, we note that the book contained the first published account of "l'Hospital's rule" along with its first published example, namely, to find the value of the quotient

$$\frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{ax^3}}$$

“when $x = a$ ” [15, p. 146]. Modern readers may be surprised to learn that the rule was framed without explicit mention of limits, a concept that still lay in the far distant future. L’Hospital’s answer to this (surprisingly complicated) initial problem was $16a/9$, which is not only correct but also indicative of how thoroughly calculus outpaces intuition.

It is well known that the rule had been discovered by Johann Bernoulli and included in the materials sent, for compensation, to l’Hospital. To the suggestion that we thus call it “Bernoulli’s rule,” the math historian Dirk Struik once scoffed, “Let the good Marquis keep his elegant rule. He paid for it” [18, p. 260].

5. EULER HALL. With that, we hurry off to the next stop: a hall devoted to the preeminent mathematician of the eighteenth century, Leonhard Euler (1707–1783).



Euler

Truth to tell, Euler was a mathematical force of nature. His collected works contain over 8,000 pages of analysis—that’s not eighty pages, nor eight hundred, but a colossal eight *thousand*. Among these are contributions to differential equations and the calculus of variations, to integrals and infinite series, and much more. Euler composed three textbooks whose influence on the subject was profound: his *Introductio in analysin infinitorum* of 1748, his *Institutiones calculi differentialis* of 1755, and the three-volume *Institutiones calculi integralis* of 1768. With such an output, it is little wonder that Euler was described as “Analysis Incarnate.”

Here we make do with a few morsels from this analytic feast. The first may have cast the longest shadow: Euler enshrined the *function* as the core idea of analysis. In the *Introductio*, he examined those functions that have been in the toolkit of analysts ever since—the logarithmic and exponential functions, trigonometric and inverse trigonometric functions, and so on. Although his 1748 definition of a function as an “analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities” is too restrictive for modern tastes, it was a marked improvement

over the ill-defined curves of his predecessors [9, p. 3]. As Euler matured, so did his notion of function, ending up closer to the modern idea of a correspondence not necessarily tied to a particular formula or “analytic expression.” It is fair to say that we now study functions in analysis because of him.

Our second Eulerian exhibit is his evaluation of

$$\int_0^1 \frac{\sin(\ln x)}{\ln x} dx,$$

the sort of calculus challenge he loved [10, vol. 18, p. 4]. Ever ready to introduce an infinite series, he applied Newton’s expansion for the sine by writing

$$\begin{aligned}\frac{\sin(\ln x)}{\ln x} &= \frac{(\ln x) - (\ln x)^3/3! + (\ln x)^5/5! - (\ln x)^7/7! + \dots}{\ln x} \\ &= 1 - \frac{(\ln x)^2}{3!} + \frac{(\ln x)^4}{5!} - \frac{(\ln x)^6}{7!} + \dots.\end{aligned}$$

This he integrated termwise to get

$$\begin{aligned}\int_0^1 \frac{\sin(\ln x)}{\ln x} dx &= \int_0^1 dx - \frac{1}{3!} \int_0^1 (\ln x)^2 dx + \frac{1}{5!} \int_0^1 (\ln x)^4 dx \\ &\quad - \frac{1}{7!} \int_0^1 (\ln x)^6 dx + \dots,\end{aligned}$$

where the integral of a sum was replaced by the sum of the integrals without missing a beat. Euler then noted that $\int_0^1 (\ln x)^n dx = n!$ for even n , so

$$\int_0^1 \frac{\sin(\ln x)}{\ln x} dx = 1 - \frac{1}{3!}[2!] + \frac{1}{5!}[4!] - \frac{1}{7!}[6!] + \dots = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}.$$

Here the Leibniz series made an unexpected appearance. This splendid derivation combined Newtonian and Leibnizian insights with a strong dose of Eulerian genius—quite a pedigree.

This is just the tip of the integration iceberg. Elsewhere in his career (see [10, vol. 19, p. 227; vol. 18, p. 8; and vol. 17, p. 407]), Euler evaluated such integrals as

$$\int_0^\infty \frac{\sin x}{x} dx, \quad \int_0^1 \frac{\sin(4 \ln x) \cos(7 \ln x)}{\ln x} dx, \quad \int_0^1 \frac{(\ln x)^5}{1+x} dx,$$

and found them to equal, respectively,

$$\frac{\pi}{2}, \quad \frac{1}{2} \arctan\left(\frac{4}{17}\right), \quad -\frac{31\pi^6}{252}.$$

No one could integrate like Uncle Leonhard!

Euler also considered foundational questions. We recall that mathematicians had had a much easier time applying calculus than explaining it. Faced with Newton’s vanishing quantities and Leibniz’s infinitely small ones, Euler thought he could see his way through the logical thicket. “An infinitely small quantity,” he wrote, “is nothing but a vanishing quantity, and so it is really equal to 0.” He continued, “There is . . . not

such a great mystery lurking in this idea as some commonly think and thereby render suspect the calculus of the infinitely small” [8, p. 51].

We can watch him apply these ideas in his treatment of l’Hospital’s rule [10, vol. 10, p. 565]. Beginning with functions P and Q for which $P(a) = Q(a) = 0$, he sought the value of $y = P/Q$ when $x = a$. Because dx was “really equal to zero,” there could be no harm in replacing a with $a + dx$. Euler thus reasoned that (recall (1))

$$y(a) = y(a + dx) = \frac{P(a + dx)}{Q(a + dx)} = \frac{P(a) + dP}{Q(a) + dQ} = \frac{0 + dP}{0 + dQ} = \frac{dP}{dQ}.$$

Voila! Among his examples of the rule was to “find the value of the expression

$$\frac{x^x - x}{1 - x + \ln(x)}$$

if we place $x = 1$.” His answer, obtained with the help of logarithmic differentiation, was -2 , a problem that remains a good one to this day.

6. THE BERKELEY ROOM. In spite of Euler’s facile explanation, the mathematical community was unconvinced that the last word had been spoken on foundational questions. This was due in no small part to the critique of a nonmathematician, the philosopher and Bishop of Cloyne, George Berkeley (1685–1763).



Berkeley

Berkeley had read justifications of the calculus and remained skeptical. He famously described Newton’s vanishing magnitudes as “the ghosts of departed quantities” and was no more charitable to Leibniz, whose infinitely small magnitudes were “above my capacity” to understand [2, p. 89 and p. 68, respectively].

To illustrate his concern, Berkeley reconsidered the standard proof that the derivative of x^n is nx^{n-1} [2, p. 72]. Mathematicians of the day began by augmenting x with a small but nonzero quantity o and looking at the ratio

$$\frac{(x+o)^n - x^n}{o}.$$

Expanding the numerator as a binomial series, they found this to equal

$$\begin{aligned} \frac{x^n + nx^{n-1}o + \frac{n(n-1)}{2}x^{n-2}o^2 + \cdots + o^n - x^n}{o} &= nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}o \\ &\quad + \cdots + o^{n-1}. \end{aligned}$$

At this point in the argument, the substitution $o = 0$ produced the derivative, nx^{n-1} .

Berkeley was having none of it. As he observed, we initially require that $o \neq 0$, “without which I should not have been able to have made so much as a single step.” Indeed, if we allow o to be zero in

$$\frac{(x+o)^n - x^n}{o},$$

not only do we have an illegitimate denominator, but x^n would never have been *augmented* in the first place. As Berkeley suggested, the whole proof would stop dead in its tracks. But we then change our minds and decide that o is zero after all. Berkeley noted that “the former supposition that the increments were something . . . is destroyed, and yet a consequence of that supposition, i.e., an expression got by virtue thereof, is retained.”

He acknowledged the correctness of the answer, but a procedure yielding correct answers from a string of mistakes was not to be trusted. “Error,” he wrote, “may bring forth truth, though it cannot bring forth science” [2, p. 77]. And he offered this caustic rejoinder [2, p. 76]:

I say that, in every other Science, Men prove their Conclusions by their Principles, and not their Principles by their Conclusions.

Clearly, foundational issues could not be long ignored in the face of such criticism. Someone had to resolve this matter, and that someone was Augustin-Louis Cauchy (1789–1857).

7. THE CAUCHY WING. Cauchy’s analytic contributions run broad and deep. Many can be found in his seminal texts, *Cours d’analyse* of 1821 and *Résumé des leçons données à l’école royale polytechnique, sur le calcul infinitésimal* of 1823. Surveying this work, math historian Carl Boyer wrote that “Cauchy did more than anyone else to impress upon the subject the character which it bears at the present time” [3, p. 271].

For starters, Cauchy believed that a rigorous treatment of calculus required neither vanishing quantities nor infinitely small ones. For him, the heart of the matter was the *limit*, and he offered this definition [6, vol. 4, p. 13]:

When the values successively attributed to a variable approach indefinitely to a fixed value, in a manner so as to end by differing from it by as little as one wishes, this last is called the limit of all the others.



Cauchy

A modern reader may be surprised by the purely verbal nature of the definition and troubled by terms like “approach” and “to end by.” Yet the insight here is solid. Cauchy perceived that, for limits, the key was to get as close as one wishes—i.e., within any preassigned target—which is the essence of the modern idea.

Equally important, Cauchy put limits front and center, building his calculus upon this foundation. Gone were proofs by picture; gone were appeals to intuition; gone were quasi-metaphysical digressions. “It would be a serious error,” he wrote, “to think that one can find certainty only in geometrical demonstrations or in the testimony of the senses” [12, p. 947].

We shall linger a while in the Cauchy Wing in order to consider two important exhibits: his proofs of the intermediate and mean value theorems.

Intermediate Value Theorem. *If f is continuous on $[x_0, X]$ with $f(x_0) < 0$ and $f(X) > 0$, then $f(a) = 0$ for some a between x_0 and X .*

Proof. Cauchy introduced $h = X - x_0$ as the width of the interval in question. Choosing a whole number $m > 1$, he noted that the values

$$f(x_0), f(x_0 + h/m), f(x_0 + 2h/m), \dots, f(x_0 + [m-1]h/m), f(X)$$

begin with a negative and end with a positive, so somewhere must appear two consecutive values of opposite sign. That is, for some n we have $f(x_0 + nh/m) \leq 0$ and $f(x_0 + [n+1]h/m) \geq 0$. Cauchy let $x_1 = x_0 + nh/m$ and $X_1 = x_0 + [n+1]h/m$ so that $x_0 \leq x_1 \leq X_1 \leq X$ and the length of subinterval $[x_1, X_1]$ is h/m .

He next divided $[x_1, X_1]$ into m equal pieces of length h/m^2 , chose two consecutive points of opposite sign, and continued, thereby generating sequences

$$x_0 \leq x_1 \leq x_2 \leq \dots \leq X_2 \leq X_1 \leq X$$

where $f(x_k) \leq 0$ and $f(X_k) \geq 0$. It was clear to him that both the nondecreasing sequence $\{x_k\}$ and the nonincreasing sequence $\{X_k\}$ converged and, because $X_k - x_k = h/m^k \rightarrow 0$, that they shared a common limit. Letting $a = \lim_{k \rightarrow \infty} x_k = \lim_{k \rightarrow \infty} X_k$, he saw that a belonged to $[x_0, X]$, and from the continuity of f he deduced that

$$f(a) = f\left[\lim_{k \rightarrow \infty} x_k\right] = \lim_{k \rightarrow \infty} f(x_k) \leq 0, \quad f(a) = f\left[\lim_{k \rightarrow \infty} X_k\right] = \lim_{k \rightarrow \infty} f(X_k) \geq 0.$$

Thus, as Cauchy put it, “ $f(a)$ cannot differ from zero” [6, vol. 3, pp. 378–380]. ■

With a small modification to tidy up the completeness property, this proof is transferable to the modern classroom. It is reminiscent of the bisection method for approximating roots of equations, but, as Judith Grabiner wrote in her treatise on Cauchy’s mathematics [11, p. 69]:

[T]hough the mechanics . . . are simple, the conception of the proof is revolutionary. Cauchy transformed the approximation technique into something entirely different: a proof of the existence of a limit.

Cauchy now turned to the mean value theorem, beginning with a preliminary result [6, vol. 4, pp. 44–46].

Lemma. *If f is continuous with A the smallest and B the largest value of f' on $[x_0, X]$, then*

$$A \leq \frac{f(X) - f(x_0)}{X - x_0} \leq B.$$

(He did not say how he knew the derivative *had* smallest and largest values.)

Proof. Cauchy began with a momentous idea. He introduced δ and ε as “very small numbers” chosen such that, for any x in $[x_0, X]$ and for any $i < \delta$, we have

$$f'(x) - \varepsilon < \frac{f(x+i) - f(x)}{i} < f'(x) + \varepsilon. \quad (2)$$

On the one hand, we celebrate the appearance of the derivative in terms of its ε - δ definition—certainly a modern touch. On the other, we recognize here the assumption of uniformity for, given an ε , Cauchy believed that “one δ fits all.” This was a misconception both subtle and pernicious.

Be that as it may, he let $x_0 < x_1 < x_2 < \dots < x_{n-1} < X$ subdivide the interval, where the distance between any two points of subdivision was less than δ . Recalling the roles of A and B and applying (2) repeatedly, he generated the inequalities:

$$A - \varepsilon \leq f'(x_0) - \varepsilon < \frac{f(x_1) - f(x_0)}{x_1 - x_0} < f'(x_0) + \varepsilon \leq B + \varepsilon,$$

$$A - \varepsilon \leq f'(x_1) - \varepsilon < \frac{f(x_2) - f(x_1)}{x_2 - x_1} < f'(x_1) + \varepsilon \leq B + \varepsilon,$$

$$A - \varepsilon \leq f'(x_2) - \varepsilon < \frac{f(x_3) - f(x_2)}{x_3 - x_2} < f'(x_2) + \varepsilon \leq B + \varepsilon,$$

⋮

$$A - \varepsilon \leq f'(x_{n-1}) - \varepsilon < \frac{f(X) - f(x_{n-1})}{X - x_{n-1}} < f'(x_{n-1}) + \varepsilon \leq B + \varepsilon.$$

At this point Cauchy wrote that “If one divides the sum of the numerators by the sum of the denominators, one obtains a mean fraction that is . . . contained between $A - \varepsilon$ and $B + \varepsilon$.” In symbols, he was saying that

$$A - \varepsilon < \frac{f(x_1) - f(x_0) + f(x_2) - f(x_1) + \cdots + f(X) - f(x_{n-1})}{x_1 - x_0 + x_2 - x_1 + \cdots + X - x_{n-1}} < B + \varepsilon,$$

which telescopes to

$$A - \varepsilon < \frac{f(X) - f(x_0)}{X - x_0} < B + \varepsilon.$$

And, “as this holds however small the number ε ,” Cauchy concluded that

$$A \leq \frac{f(X) - f(x_0)}{X - x_0} \leq B.$$
■

He then was ready for one of the great theorems of differential calculus.

Mean Value Theorem. *If f and f' are continuous between x_0 and X , then for some θ between 0 and 1, it is the case that*

$$\frac{f(X) - f(x_0)}{X - x_0} = f'[x_0 + \theta(X - x_0)].$$

Before giving his proof, we make two observations. First, because $0 < \theta < 1$, the number $x_0 + \theta(X - x_0)$ lies strictly between x_0 and X and so plays the role of “ c ” in modern statements of the mean value theorem.

Second, Cauchy assumed the continuity of f' . As we shall see, he did so to ensure that the derivative possess the intermediate value property. A later theorem of Gaston Darboux (1842–1917) would establish that, even if discontinuous, f' must exhibit this property, so Cauchy did not need his assumption. (Nowadays, of course, we prove the mean value theorem quite differently, without mentioning the continuity of f' at all.)

Proof. The lemma guarantees that

$$\frac{f(X) - f(x_0)}{X - x_0}$$

is intermediate between A and B , the minimum and maximum values of f' on $[x_0, X]$. By the assumed continuity of the derivative, f' takes this intermediate value somewhere between x_0 and X . In other words, there is a θ between 0 and 1 with

$$f'[x_0 + \theta(X - x_0)] = \frac{f(X) - f(x_0)}{X - x_0}.$$
■

Thanks to Cauchy, the calculus had come a long way.

8. TRANSITION TO THE MODERN WING. The following half century saw three developments that carried the subject into ever more sophisticated realms. We mention each briefly as we proceed to the next major exhibit.

1. *Mathematicians refined the fundamental ideas of analysis.* By the time of Karl Weierstrass (1815–1897), the limit definition had been distilled to:

Definition. $\lim_{x \rightarrow a} f(x) = L$ if and only if, for each $\varepsilon > 0$, there exists a $\delta > 0$ so that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

This string of well-chosen inequalities gave an unambiguous and purely static meaning to limits. In the hands of the next generation of mathematicians, it allowed for new standards of analytic rigor.

There were other matters in need of attention. It was Weierstrass who emphasized the distinction between continuity and uniform continuity and between pointwise and uniform convergence. And Georg Friedrich Bernhard Riemann (1826–1866) developed his integral, in the process forever separating the ideas of continuity and integrability. So natural and powerful was his approach that the Riemann integral seemed incapable of improvement.

2. *Pathological counterexamples proliferated.* These were sufficiently bizarre as to defy belief, yet their existence forced mathematicians to think more deeply about critical ideas. For instance, Weierstrass described the seemingly impossible: an everywhere continuous, nowhere differentiable function [20, vol. 2, pp. 71–74]. His original example, rarely found in modern texts, was $f(x) = \sum_{k=0}^{\infty} b^k \cos(\pi a^k x)$, where

- a is an odd integer greater than one
- $0 < b < 1$
- $ab > 1 + \frac{3\pi}{2}$

(These side conditions, as unexpected as they were essential to Weierstrass's reasoning, are sure signs of an analyst at work.) Any partial sum of the series yields a function everywhere continuous and everywhere differentiable, yet when we pass to the limit, the result is everywhere continuous but *nowhere* differentiable. So much for intuition.

Another provocative counterexample came from Vito Volterra (1860–1940). In 1881 he described a function that was differentiable with a bounded derivative, yet whose derivative was so wildly discontinuous that its (Riemann) integral did not exist [19, pp. 16–48]. His example thus destroyed an “unqualified” fundamental theorem of calculus. That is, if f is differentiable with f' bounded on $[a, b]$, then we would hope that

$$\int_a^b f'(x) dx = f(x) \Big|_a^b.$$

Volterra's function invalidated this equation, not because the left side failed to equal the right side but because the left side failed even to *exist* as a Riemann integral.

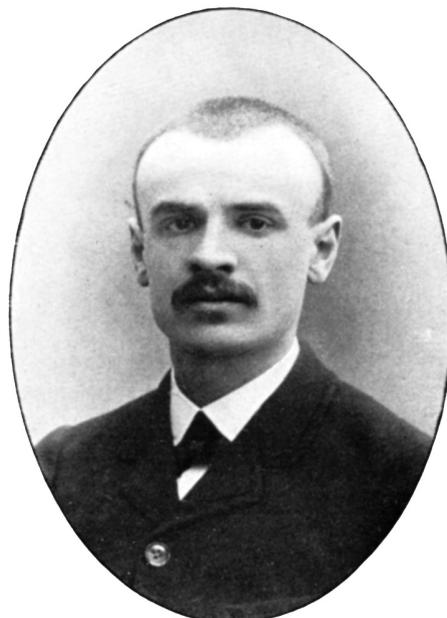
These and other pathological counterexamples raised significant questions, among which were:

- How discontinuous can an integrable function be?
- How discontinuous can a derivative be?
- How, if at all, can the fundamental theorem of calculus be repaired?

We shall have more to say about each of these before our visit to the Calculus Gallery is over.

3. *Set theory arrived as an ally of the analyst.* The innovator here was Georg Cantor (1845–1918). Highlighting his initial paper on this subject from 1874 was a proof that a sequence of real numbers cannot exhaust an open interval [5, pp. 115–118]. In modern parlance, this established the nondenumerability of the continuum, and one might naturally assume that Cantor used diagonalization as his weapon of choice. That line of attack, however, dates to 1891. His original argument invoked the completeness property of the reals and thus lay squarely in the analytic domain. Employing sets in the service of analysis would prove to be a fruitful undertaking, as Weierstrassian rigor fused with Cantorian set theory to create the subject we know today.

9. THE BAIRE EXHIBIT. One of the first to move in this direction was René Baire (1874–1932). Baire, who admired Cantor and had studied with Volterra, took their lessons to heart. His 1899 dissertation *Sur les fonctions de variables réelles* contained what has come to be known as the Baire category theorem, a featured exhibit of the Calculus Gallery.



Baire

Baire began by defining a set P of real numbers to be *nowhere dense* if every open interval (α, β) has an open subinterval (a, b) containing no point of P . Examples include all finite sets as well as the set $\{1/n : n \text{ is a whole number}\}$. He then wrote [1, p. 65]:

If there exists a denumerable infinity of nowhere dense sets P_1, P_2, P_3, \dots such that every point of F belongs to at least one of them, ... I shall say that F is of the *first category*.

In coining so nondescriptive a term—“first category,” after all, conjures up absolutely nothing in the imagination—Baire earned a reputation for colorless terminology. This was cemented when he gave all other sets the moniker “second category.”

If his terms were bland, his mathematics was not. In that 1899 thesis, Baire proved the result that carries his name:

Theorem. *A first-category set cannot exhaust an open interval.*

Proof. Begin with an open interval (α, β) and let $F = P_1 \cup P_2 \cup P_3 \cup \dots$ be a set of the first category, where each P_k is nowhere dense. Because P_1 is nowhere dense, there is an open subinterval of (α, β) containing no points of P_1 . By shrinking the subinterval if necessary, we can find points a_1 and b_1 with $a_1 < b_1$ such that $[a_1, b_1] \subseteq (\alpha, \beta)$ and $[a_1, b_1] \cap P_1 = \emptyset$.

But (a_1, b_1) is open and P_2 is nowhere dense, so there exists a_2 and b_2 with $a_2 < b_2$ such that $[a_2, b_2] \subseteq (a_1, b_1)$ and $[a_2, b_2] \cap P_2 = \emptyset$. Repeating the argument, Baire generated descending intervals

$$[a_1, b_1] \supseteq [a_2, b_2] \supseteq \dots \supseteq [a_k, b_k] \supseteq \dots,$$

where $[a_k, b_k] \cap P_k = \emptyset$ for each k . By completeness, there is a point c common to all these closed intervals. Thus $c \in (\alpha, \beta)$ but $c \notin P_1 \cup P_2 \cup P_3 \cup \dots = F$. In short, a first category set is insufficient to fill an open interval. ■

Two remarks are in order. First, the debt to Cantor is evident. As we noted, Cantor had proved that a denumerable set cannot exhaust an open interval, and Baire extended this principle to any first category set (of which denumerable sets are special cases). Second, the theorem's conclusion could be recast to say that the complement of a first category set is dense, for points of this complement must show up in any open interval.

With these powerful ideas, Baire was able to answer one of our earlier questions: How discontinuous can a derivative be? By proving that the discontinuity points of a derivative form a set of the first category, he concluded:

Corollary. *If f is differentiable on an interval I , then f' must be continuous on a dense subset of I .*

Thus, derivatives can be discontinuous, but not so discontinuous that there is an open interval free of their continuity points.

René Baire was among the new wave of analysts, active at the turn of the twentieth century, who viewed the subject through the lens of set theory. He asserted (see [1, p. 121]) that

[A]ny problem relative to the theory of functions leads to certain questions relative to the theory of sets and, insofar as these latter questions are or can be addressed, it is possible to resolve, more or less completely, the given problem.

10. THE LEBESGUE ROOM. With that pronouncement as background, we enter the gallery's final room, one devoted to Baire's great contemporary, Henri Lebesgue (1875–1941). Lebesgue's analytic reputation was established with his 1904 classic *Leçons sur l'intégration*, a work featuring a bewildering array of seminal ideas.

He began with a survey of the Riemann integral, reviewing its history and noting those shortcomings described earlier. In response, Lebesgue defined a set to have *measure zero* if, in his words, it “can be enclosed in a finite or a denumerable infinitude of intervals whose total length is as small as we wish” [13, p. 28]. Then, in an ingenious argument using the Heine-Borel theorem, he proved that a bounded function on



Lebesgue

$[a, b]$ is Riemann integrable over $[a, b]$ if and only if its set of discontinuities on $[a, b]$ has measure zero. This resolved the question as to how discontinuous a Riemann integrable function can be: in terms of measure, the answer is “not very.” With this result, Henri Lebesgue could lay claim to having understood the *Riemann* integral more fully than anyone. In light of what was to come, this held a certain irony.

Lebesgue next developed a theory of measure for subsets of the real line—not just for those of measure zero. From there, in a manner used to this day, he explored the notion of a *measurable function* and then moved to his most famous creation: the Lebesgue integral.

Obviously, he had studied Riemann well. Lebesgue acknowledged the debt by noting that his new integral [14, p. 136]

is analogous to that of Riemann; but whereas he divided into small subintervals the interval of variation of x , it is the interval of variation of $f(x)$ that we have subdivided.

In short, partition the range not the domain, and that will make all the difference.

Lebesgue carried the theory far enough to convince him of the superiority of his integral over Riemann’s. For instance, in what we now call the bounded convergence theorem, he proved:

Theorem. *If measurable functions f_k ($k = 1, 2, \dots$) are uniformly bounded on $[a, b]$ and have pointwise limit f on this interval, then*

$$\lim_{k \rightarrow \infty} \left[\int_a^b f_k(x) dx \right] = \int_a^b f(x) dx = \int_a^b \left[\lim_{k \rightarrow \infty} f_k(x) \right] dx.$$

This spectacular result says that, under minimal restrictions, the limit of the (Lebesgue) integrals is the (Lebesgue) integral of the limit [13, p. 114]. The theorem fails for Riemann's integral.

But there was more. Lebesgue established that, with *his* integral, the fundamental theorem of calculus could be salvaged as follows [13, p. 120]:

Theorem. *If f is differentiable on $[a, b]$ with bounded derivative, then*

$$\int_a^b f'(x) dx = f(x) \Big|_a^b.$$

This repaired the damage inflicted by Volterra's counterexample. In that sense it restored the fundamental theorem of calculus to its "natural" state, so long as we adopt Lebesgue's version of the integral. One hopes that Newton and Leibniz, from their simpler vantage points, would have been pleased that their creation had returned to a kind of analytic nirvana.

And that, perhaps, is a fitting way to end our visit to the Calculus Gallery. As is customary, we pop into the Museum Shop with its array of scarves, jewelry, and other noteworthy items, chief among which are reproductions of the original texts. Anyone interested in primary sources is urged to consult the works of Newton, Euler, Cauchy, or Lebesgue. Their achievements seem magnified when encountered in original form.

For those interested in secondary sources, the shop carries three books that come highly recommended: Ivor Grattan-Guinness's *The Development of the Foundations of Mathematical Analysis from Euler to Riemann* (MIT Press, 1970), Judith Grabiner's *The Origins of Cauchy's Rigorous Calculus* (MIT Press, 1981), and Thomas Hawkins's classic *Lebesgue's Theory of Integration* (Chelsea, 1975). Each is an excellent read.

With a bag full of new purchases, we are ready to go. Approaching the exit, we notice one more inscription on the rotunda walls. As before, it is due to von Neumann, and we leave the Calculus Gallery with his words in our minds [16, p. 3]:

I think it [the calculus] defines more unequivocally than anything else the inception of modern mathematics, and the system of mathematical analysis, which is its logical development, still constitutes the greatest technical advance in exact thinking.

ACKNOWLEDGMENT. This article is based upon "Selections from the Calculus Museum," an invited MAA address delivered at the 2004 Joint Meetings in Phoenix.

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