Linear Algebra, a Potent Tool

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1. Introduction. Most college courses in linear algebra overwhelm students, for the first time, with abstractions. They have to learn definitions of "linear independence," "basis," "dimension," "invariant subspace," etc., and their instructors rarely have time to present applications and to demonstrate the power of these methods.

A gentler path to linear algebra would begin with two dimensions. The ideal space, for this purpose, is the ordinary Euclidean plane, and the fruits of this labor are theorems in plane geometry; the study of bi-linear and quadratic forms can be applied to conic sections. In addition to preparing people for linear algebra in n dimensions, such a course would have the virtue of teaching all the content usually found in an analytic geometry course and in many elementary synthetic geometry courses.

There are many other uses of linear algebra. I want to present one of these in this article. Only the tools of the two-dimensional trade will be needed.

We shall use linear algebra to study sequences defined by linear recursion relations. We begin by describing some examples.

2. Examples. (i) Assume rabbits breed in such a way that, when a pair reaches the age of 2 months, it begets another pair every month. Assume, moreover, that our rabbits are immortal. If a single newborn pair of ancestor rabbits begins this breeding process, how many pairs of rabbits will there be in the nth month?

During the first month, there is 1 pair; during the second, still 1 pair; during the third, 2 pairs: the ancestor and one offspring pair. During the third month, there will be 3 pairs (the ancestor, last month's offspring, and a new offspring produced by the ancestors). We observe that during the nth month, we have all pairs present during the $(n-1)$st month and, in addition, a new offspring from each pair that existed during the $(n-2)$nd month. Thus, denoting the number of pairs present during the nth month by f_n, we find that

$$f_n = f_{n-2} + f_{n-1}, \qquad n = 3, 4, 5, \ldots . \tag{1}$$

Equation (1) gives each term of the sequence, from the third on, as a linear combination of its predecessors. Such a formula is called a *linear recursion relation.*

Together with the information $f_1 = f_2 = 1$, it enables us to find all subsequent terms f_i $(i = 3, 4, \dots)$ of this sequence, called the Fibonacci sequence, and appearing in a number of phenomena.[1]

The methods we shall describe will yield a formula for f_n directly*, so that the f_i for $i < n$ need not be computed first.

(ii) As a second example, consider two gamblers, $\mathcal{A}$ and $\mathcal{B}$, playing the following game: A coin is tossed. If it lands head up, $\mathcal{A}$ gets a dollar from $\mathcal{B}$; if it lands tail up, $\mathcal{A}$ loses a dollar to $\mathcal{B}$. The gamblers start with d dollars of which $\mathcal{A}$ has i and $\mathcal{B}$ has d-i. The game ends when one of the players runs out of money; that is, when he is ruined. We wish to find the probability a_i that $\mathcal{A}$ is ruined if he starts with i dollars, for $i = 0, 1, 2, \dots, d$.

If the coin is such that the probability of heads is p, then that of tails is $q = 1 - p$. If the coin is fair, $p = q = 1/2$.

Now, suppose $\mathcal{A}$ has i dollars. One of two (mutually exclusive) things must happen at the next toss:

(a) the coin turns up heads, so $\mathcal{A}$ will have $i + 1$ dollars,
(b) the coin turns up tails, so $\mathcal{A}$ will have $i - 1$ dollars.

Thus, the probability of $\mathcal{A}$'s ruin when he has i dollars is the sum of

(a) the probability of {heads, followed by $\mathcal{A}$'s ruin if he has $(i + 1)$ dollars} = pa_{i+1}

and

(b) the probability of {tails, followed by $\mathcal{A}$'s ruin if he has $(i - 1)$ dollars} = qa_{i-1}.

Thus

$$a_i = pa_{i+1} + qa_{i-1}, \qquad i = 1, 2, 3, \dots, d - 1.^{**} \tag{2}$$

Moreover, $a_0 = 1$ because when $\mathcal{A}$ has no money, his ruin is a certainty; and $a_d = 0$, because then $\mathcal{B}$ has been ruined, and $\mathcal{A}$ cannot be ruined.

Let us solve (2) for a_{i+1} in terms of its two predecessors, a_{i-1} and a_i. We obtain the linear recursion formula

$$a_{i+1} = -(q/p)a_{i-1} + (1/p)a_i, \qquad i = 1, 2, \dots, d - 1. \tag{3}$$

Using (3), the values $a_0 = 1$, $a_d = 0$, and the given probabilities p, q (with $p + q = 1$), we wish to calculate a_i in terms of the total capital d and $\mathcal{A}$'s initial share of it, i.***

[1] See, for example, H. S. M. Coxeter, *Introduction to Geometry*, Wiley, New York, 1961, p. 165.

* This was done also by Paul Garrett in *Matrix Eigenvalues: Characteristic Values*, the Mathematics Student Journal, May 1971, p. 4; he was then a high school student.

** For a more detailed derivation of (3), see, e.g., *The Elements of Probability* by Simeon M. Berman, Addison-Wesley, 1969, pp. 77–84.

*** In the case $p = q = \frac{1}{2}$, this problem was solved in *Mathematical Gems* by R. Honsberger, Dolciani Mathematical Expositions, vol. 1, MAA, 1973, pp. 128–130.

(iii) Our third example does not involve recursion relations from "real life"; instead they were man-made for a problem of the 1973 U.S.A. Mathematical Olympiad.* We shall sketch its solution later on in this article.

The problem defines two sequences $\{x_n\}$ and $\{y_n\}$ of integers by

$$x_0 = 1, x_1 = 1 \qquad x_{n+1} = 2x_{n-1} + x_n \qquad n = 1, 2, \ldots \tag{4}$$

$$y_0 = 1, y_1 = 7 \qquad y_{n+1} = 3y_{n-1} + 2y_n \qquad n = 1, 2, \ldots \tag{5}$$

and asks the student to prove that, except for the term "1", the two sequences have no term in common.

3. General principles. The central feature of our examples is a recursion formula of the form

$$x_{n+1} = ax_{n-1} + bx_n, \qquad n = 1, 2, \ldots \tag{6}$$

where a, b are given constants. We view each pair of consecutive terms of the sequence x_0, x_1, x_2, ... as components of a two-dimensional vector:

$$X_0 = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}, \quad X_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ \ldots, \quad X_n = \begin{pmatrix} x_n \\ x_{n+1} \end{pmatrix}, \ldots; \tag{7}$$

and we use the recursion formula to determine a linear transformation T which maps each vector into the next: $TX_i = X_{i+1}$. This transformation is given by the coefficient matrix of the pair of equations

$$\begin{aligned} x_n &= \phantom{ax_{n-1} + b} x_n \\ x_{n+1} &= ax_{n-1} + bx_n . \end{aligned}$$

This is equivalent to the single vector equation

$$\begin{pmatrix} x_n \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix} \begin{pmatrix} x_{n-1} \\ x_n \end{pmatrix} \quad \text{or} \quad X_n = TX_{n-1}, \tag{8}$$

where $T = \left(\begin{smallmatrix} 0 & 1 \\ a & b \end{smallmatrix}\right)$. We obtain the desired sequence by iterating T:

$$X_1 = TX_0, X_2 = TX_1 = T(TX_0) = T^2X_0, \ldots, X_n = T^nX_0, \ldots. \tag{9}$$

We see that X_n is expressed directly in terms of X_0 and the nth power of the matrix T. If we actually had to compute successive powers T^2, T^3, $\cdots$ of the matrix T, we would be no better off than we were when we computed successive terms x_i of the sequence via the recursion formula. It is at this point that linear algebra comes to the rescue with a simple way of finding T^nX_0, provided we express X_0 as linear combination of eigenvectors of T. We recall that an *eigenvector* E of T is a vector transformed by T into a multiple of itself:

$$TE = \lambda E; \tag{10}$$

* *The Second USA Mathematical Olympiad* by S. L. Greitzer, the Mathematics Teacher, Feb. 1974, p. 115, contains statements and brief solutions to all five problems of this contest.

the *eigenvalue* λ is the factor by which T stretches or shrinks E. Consequently,

$$T^2E = T(TE) = T(\lambda E) = \lambda TE = \lambda^2 E,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$T^nE = \lambda^n E.$$

If we can find two linearly independent eigenvectors $E^{(1)}$ and $E^{(2)}$ of T, then we can express any vector X_0 in the form

$$X_0 = c_1E^{(1)} + c_2E^{(2)}$$

and compute

$$X_n = T^nX_0 = c_1\lambda_1{}^nE^{(1)} + c_2\lambda_2{}^nE^{(2)}, \tag{11}$$

where λ_1, λ_2 are the eigenvalues corresponding to the eigenvectors $E^{(1)}$, $E^{(2)}$, respectively. Formula (11) allows us to compute X_n directly, without first having to find X_i for $i < n$. In addition, it furnishes much information about our series for large n.

To find eigenvalues λ and eigenvectors E, we must solve equation (10) or, equivalently,

$$TE - \lambda E = (T - \lambda I)E = 0, \tag{12}$$

where I is the identity matrix $\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$. The matrix $T - \lambda I$ must annihilate E, but in order to map a non-zero vector into the zero vector, a matrix must be singular; that is, its determinant must be 0. This requirement leads to the characteristic equation

$$|T - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ a & b-\lambda \end{vmatrix} = \lambda^2 - \lambda b - a = 0, \tag{12'}$$

a quadratic for λ with solutions

$$\lambda_1 = (b + \sqrt{b^2+4a})/2 \quad \text{and} \quad \lambda_2 = (b - \sqrt{b^2+4a})/2,$$

which are real whenever $b^2 \geq -4a$ and real and distinct whenever $b^2 > -4a$.

Once the eigenvalues are determined, we can find the eigenvectors $E^{(1)}$ and $E^{(2)}$ by solving equation (10) for E using $\lambda = \lambda_1$, and $\lambda = \lambda_2$, respectively. We now apply these principles to our examples.

4. Application to Example (i). For the Fibonacci sequence, the transformation matrix T:

$$\begin{pmatrix} f_{n-2} \\ f_{n-1} \end{pmatrix} \rightarrow \begin{pmatrix} f_{n-1} \\ f_n \end{pmatrix} \quad \text{is} \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

since

$$f_{n-1} = 0\cdot f_{n-2} + 1\cdot f_{n-1}$$
$$f_n = 1\cdot f_{n-2} + 1\cdot f_{n-1}.$$

[Although this sequence usually "begins" 1, 1, 2, 3, 5, ... $= f_1, f_2, \ldots$ so that

$$F_1 = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

we may augment it to 1, 0, 1, 1, 2, ... $= x_0, x_1, \ldots$, so that the first two vectors,

$$X_0 = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad X_1 = TX_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

are the basis vectors, and

$$X_n = T^n X_0 = F_{n-2} = T^{n-2} F_1 \,.]$$

We can now solve the characteristic equation for T to obtain its eigenvalues and eigenvectors, or we can find these directly by solving equation (10). Since every multiple of an eigenvector is again an eigenvector, we may normalize our eigenvector (to simplify later computation) to the form $E = \binom{e}{1}$; then equation (10) becomes

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ e+1 \end{pmatrix} = \begin{pmatrix} \lambda e \\ \lambda \end{pmatrix},$$

or

$$1 = \lambda e, \qquad e + 1 = \lambda. \tag{13}$$

By eliminating e, we obtain the characteristic equation $\lambda^2 - \lambda - 1 = 0$ with solutions

$$\lambda_1 = (1 + \sqrt{5})/2, \qquad \lambda_2 = (1 - \sqrt{5})/2, \tag{14}$$

satisfying

$$\lambda_1 + \lambda_2 = 1, \qquad \lambda_1\lambda_2 = -1. \tag{15}$$

To find the eigenvectors, we determine e from (13), and obtain $e = \lambda - 1$ so that

$$E^{(1)} = \begin{pmatrix} \lambda_1 - 1 \\ 1 \end{pmatrix}, \qquad E^{(2)} = \begin{pmatrix} \lambda_2 - 1 \\ 1 \end{pmatrix}$$

or, in view of the first relation in (15),

$$E^{(1)} = \begin{pmatrix} -\lambda_2 \\ 1 \end{pmatrix}, \qquad E^{(2)} = \begin{pmatrix} -\lambda_1 \\ 1 \end{pmatrix}.$$

[We note, in passing, that the eigenvectors are orthogonal, i.e., $E^{(1)} \cdot E^{(2)} = \lambda_1\lambda_2 + 1 = -1 + 1 = 0$; this is always true of the eigenvectors of a symmetric matrix.]

Next, we express X_0 as a linear combination of eigenvectors:

$$X_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 E^{(1)} + c_2 E^{(2)} = \begin{pmatrix} -c_1\lambda_2 - c_2\lambda_1 \\ c_1 + c_2 \end{pmatrix}.$$

The coefficients c_1, c_2 satisfy $c_1 + c_2 = 0$ and $-c_1\lambda_2 - c_2\lambda_1 = 1$, whence $c_1 = 1/(\lambda_1 - \lambda_2) = 1/\sqrt{5}$, $c_2 = -c_1 = -1/\sqrt{5}$, and

$$X_n = T^n X_0 = c_1\lambda_1^n E^{(1)} + c_2\lambda_2^n E^{(2)} = \frac{1}{\sqrt{5}}\begin{pmatrix} -\lambda_1^n\lambda_2 + \lambda_1\lambda_2^n \\ \lambda_1^n - \lambda_2^n \end{pmatrix}. \tag{16}$$

In view of the second relation in (15), this yields

$$X_n = \frac{1}{\sqrt{5}}\begin{pmatrix} \lambda_1^{n-1} - \lambda_2^{n-1} \\ \lambda_1^n - \lambda_2^n \end{pmatrix}. \tag{17}$$

Thus the nth term of the Fibonacci sequence is

$$f_n = x_{n+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]. \tag{18}$$

This is the formula we were seeking.

What other interesting information does equation (16) provide? We re-write it in the form

$$X_n = \frac{\lambda_1^n}{\sqrt{5}}\left[E^{(1)} - \left(\frac{\lambda_2}{\lambda_1}\right)^n E^{(2)}\right],$$

and we observe that, since $|\lambda_1| > |\lambda_2|$ ($\lambda_1 \approx 1.616$, $\lambda_2 \approx -.618$), the component of X_n in the eigendirection $E^{(1)}$ dominates, so that X_n approaches that direction as $n \to \infty$. The fact that λ_2 is negative causes the sequence of vectors X_n to oscillate about the limiting direction $E^{(1)}$. Since the direction of a vector is determined by the ratio of its second to its first component, it follows that

$$\frac{\text{2nd component of } X_n}{\text{1st component of } X_n} = \frac{x_{n+1}}{x_n} = r_n \text{ approaches } \frac{\text{2nd component of } E^{(1)}}{\text{1st component of } E^{(1)}}$$

$$= \frac{1}{-\lambda_2} = \lambda_1.$$

In other words, the ratio r_n of the $(n+1)$st to the nth term of the Fibonacci sequence approaches the limit $(1+\sqrt{5})/2$ as $n \to \infty$. Successive ratios r_n approach λ_1 alternately from above and below.

5. Application to Example (ii). The transformation matrix in the gambler's ruin is

$$T = \begin{pmatrix} 0 & 1 \\ -q/p & 1/p \end{pmatrix}, \qquad p + q = 1;$$

its characteristic equation is $\lambda^2 - (1/p)\lambda + q/p = 0$ and has discriminant $1/p^2$ times $1 - 4pq = 1 - 4p(1-p) = 1 - 4p + 4p^2 = (1-2p)^2$, which is never negative, and vanishes if and only if $p = 1/2$. Consequently, the eigenvalues

$\lambda_1 = (1/2p)[1 + (1 - 2p)] = (1 - p)/p = q/p$ and $\lambda_2 = (1/2p)[1 - (1 - 2p)] = 1$ are always real; they are equal if and only if the tossed coin is fair ($p = q = 1/2$). The first eigenvector

$$E^{(1)} = \begin{pmatrix} 1 \\ e^{(1)} \end{pmatrix}$$

must satisfy $TE^{(1)} = \lambda_1 E^{(1)}$, or

$$\begin{pmatrix} 0 & 1 \\ -q/p & 1/p \end{pmatrix}\begin{pmatrix} 1 \\ e^{(1)} \end{pmatrix} = \begin{pmatrix} e^{(1)} \\ \dfrac{-q + e^{(1)}}{p} \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_1 e^{(1)} \end{pmatrix},$$

so $e^{(1)} = \lambda_1 = q/p$, and

$$E^{(1)} = \begin{pmatrix} 1 \\ q/p \end{pmatrix}.$$

For the second eigenvector,

$$E^{(2)} = \begin{pmatrix} 1 \\ e^{(2)} \end{pmatrix},$$

$$TE^{(2)} = \begin{pmatrix} e^{(2)} \\ \dfrac{-q + e^{(2)}}{p} \end{pmatrix} = \begin{pmatrix} \lambda_2 \\ \lambda_2 e^{(2)} \end{pmatrix} = \begin{pmatrix} 1 \\ e^{(2)} \end{pmatrix} = E^{(2)},$$

so $e^{(2)} = 1$ and $E^{(2)} = \binom{1}{1}$.

Next, our method calls for the representation of X_0 as linear combination of $E^{(1)}$ and $E^{(2)}$, so that we may compute

$$X_n = T^n X_0 = T^n[c_1 E^{(1)} + c_2 E^{(2)}] = c_1 \lambda_1^n E^{(1)} + c_2 \lambda_2^n E^{(2)}.$$

In our previous example both components of the initial vector X_0 were known and furnished the pair of equations for determining c_1 and c_2. In the present example, only the first component $x_0 = a_0 = 1$ of X_0 is known (see p. 4), but we also know the second component $x_d = a_d = 0$ of X_{d-1} (see p. 4), where d is the combined capital of our gamblers. So this time, we get our pair of equations for c_1, c_2 from the first component of

$$X_0 = \begin{pmatrix} 1 \\ x_1 \end{pmatrix} = c_1 E^{(1)} + c_2 E^{(2)} \tag{19}$$

and the second component of

$$X_{d-1} = \begin{pmatrix} x_{d-1} \\ 0 \end{pmatrix} = c_1 \lambda_1^{d-1} E^{(1)} + c_2 \lambda_2^{d-1} E^{(2)}, \tag{20}$$

using the eigenvalues and eigenvectors found above. These equations for c_1 and c_2 are

$$1 = c_1 + c_2$$

$$0 = c_1 z^d + c_2, \qquad \text{where } z = q/p = \lambda_1,$$

and yield $c_1 = 1/(1 - z^d)$, $c_2 = (z^d)/(z^d - 1)$.
Consequently

$$X_i = c_1 \lambda_1{}^i E^{(1)} + c_2 \lambda_2{}^i E^{(2)} = \frac{1}{1 - z^d}\begin{pmatrix} z^i - z^d \\ z^{i+1} - z^d \end{pmatrix}.$$

The formula for the probability of $\mathcal{A}$'s ruin if $\mathcal{A}$ starts with i dollars is

$$a_i = x_i = \frac{z^i - z^d}{1 - z^d} = \frac{(q/p)^i - (q/p)^d}{1 - (q/p)^d}, \qquad i = 0, 1, 2, \ldots, d. \tag{21}$$

The value of a_i can be directly computed provided the properties of the coin, that is the probability of heads, is known, and the coin is not fair. If the coin is fair, $z = q/p = 1$ and formula (21) cannot be used.

To analyze the probability of $\mathcal{A}$'s ruin with a fair coin, we can proceed in two ways. We can examine the behavior of $a_i(z) = (z^i - z^d)/(1 - z^d)$ as z approaches 1, and see if $\lim_{z\to 1} a_i(z)$ exists. This is an analytic approach. Alternatively, we can use the theory of linear algebra to study the transformation matrix $T = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$ with the double eigenvalue 1 and with only one eigendirection $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. We shall do each in turn; both methods work and yield the same result.

We re-write $a_i(z)$ in the form

$$a_i(z) = \frac{z^d - z^i}{z^d - 1} = \frac{x^{(d-i)/d} - 1}{x - 1} = b_i(x)$$

where $x = z^{-d}$; and as $z \to 1$, $x \to 1$. Now set $f(x) = x^{(d-i)/d}$ and let $h = x - 1$ so that

$$f(1 + h) = f(x) = x^{(d-i)/d}, \qquad f(1) = 1,$$

and

$$\lim_{x\to 1} b_i(x) = \lim_{h\to 0} \frac{f(1 + h) - f(1)}{h} = f'(1) = \frac{d - i}{d} = \lim_{z\to 1} a_i(z). \tag{22}$$

Thus for a fair coin, (22) yields the probabilities

$$a_0 = 1,\ a_1 = \frac{d - 1}{d},\ a_2 = \frac{d - 2}{d},\ \ldots,\ a_{d-1} = \frac{1}{d},\ a_d = 0.$$

Our algebraic method yields the transformation matrix

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \qquad \text{for} \quad p = q = \tfrac{1}{2},$$

with equal eigenvalues $\lambda_1 = \lambda_2 = 1$. All solutions of $TE = \lambda E$, i.e., of

$$\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} e_2 \\ -e_1 + 2e_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

are multiples of the vector $\binom{1}{1}$, so we take

$$E = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

as our eigenvector. We recall from linear algebra that, if a matrix fails to have a full set of linearly independent eigenvectors (two in our case), we may introduce a generalized eigenvector (a vector mapped into a multiple of itself by T^2). In our case any vector not in the direction of E would serve; but the most convenient for our purposes is a vector G for which

$$TG = G + E; \tag{23}$$

for then

$$T^2G = TG + TE = G + 2E,\ T^3G = G + 3E,\ \ldots,\ T^nG = G + nE,$$

and if $X_0 = c_1E + c_2G$, then

$$X_n = T^nX_0 = c_1E + c_2(G + nE) = (c_1 + nc_2)E + c_2G, \tag{24}$$

still a simple expression. Solving (23) is equivalent to solving $(T - I)G = E$. In other words, since we cannot find a vector independent of E annihilated by $T - I$, we shall at least find one which is mapped into E by $T - I$, hence annihilated by $(T - I)^2$. We solve

$$(T - I)G = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = E \qquad \text{for} \quad G = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix};$$

$-g_1 + g_2 = 1$, $g_2 = 1 + g_1$. Take $G = \binom{1}{2}$. Now the analogues of equations (19) and (20) are, in view of (24),

$$X_0 = c_1E + c_2G, \tag{25}$$

$$X_{d-1} = T^{d-1}X_0 = c_1E + c_2(G + (d - 1)E) = [c_1 + (d - 1)c_2]E + c_2G. \tag{26}$$

Again, we determine c_1 and c_2 from the first component of X_0 and the second of X_{d-1}:

$$1 = c_1 + c_2, \qquad 0 = c_1 + (d - 1)c_2 + 2c_2 = c_1 + (d + 1)c_2,$$

so $c_1 = (d + 1)/d$, $c_2 = -1/d$, $c_1 + nc_2 = (d + 1 - n)/d$;

$$X_i = \frac{d + 1 - i}{d}E - \frac{1}{d}G,$$

and

$$x_i = a_i = \frac{d - i}{d}$$

as in the analytic result.

6. Application to Example (iii). Application of our methods to the sequences of the Olympiad problem lead to its solution. We shall give only the broad outlines and leave the details to the interested reader.

The transformation matrices for the series $\{x_i\}$ and $\{y_i\}$ defined by (4) and (5) are

$$T_x = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad T_y = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix},$$

respectively. The eigenvalues and corresponding eigenvectors of T_x are

$$\begin{matrix} \lambda_1 = -1, \\ \lambda_2 = 2, \end{matrix} \qquad E^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad E^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

and those of T_y are

$$\begin{matrix} \mu_1 = -1, \\ \mu_2 = 3 \end{matrix} \qquad F^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad F^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

These lead to the representations

$$X_0 = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \tfrac{1}{3} E^{(1)} + \tfrac{2}{3} E^{(2)}, \; Y_0 = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = -F^{(1)} + 2F^{(2)}$$

and

$$x_n = \tfrac{1}{3}[(-1)^n + 2^{n+1}], \qquad y_k = (-1)^{k+1} + 2 \cdot 3^k.$$

For $n = k = 0$, $x_0 = y_0 = 1$. To see that $x_n \neq y_k$ for all positive integers n and k, we must show that $x_n = y_k$, or equivalently,

$$2[3^{k+1} - 2^n] = (-1)^n + 3(-1)^k,$$

holds only if $n = k = 0$. This can easily be demonstrated, for example, by considering separately the cases: (1) n and k are both even, (2) n is odd and k is even, (3) n is even and k is odd, (4) n and k are both odd. Somewhat different solutions are sketched in the article quoted in the footnote of p. 5.

7. Extension. The methods we have applied here can easily be extended to treat linear recursion formulas of the form

$$x_n = a_k x_{n-k} + \cdots + a_2 x_{n-2} + a_1 x_{n-1},$$

where each term of a sequence is expressed as a linear combination of its k predecessors. This would lead to a k-dimensional vector space and a $k \times k$ transformation matrix

$$T = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & 1 \\ a_k & a_{k-1} & \cdot\cdot & a_2 & a_1 \end{pmatrix}.$$

8. Generalization. We conclude this article by giving another interpretation of recursion relations from the point of view of abstract linear algebra. This new interpretation is suitable also for the study of differential equations with constant coefficients. We shall sketch it first for recursion relations of the form (6), then give its analogue for second order differential equations.

To this end, we view sequences $x_1, x_2, x_3, \ldots$ as *functions* F defined on the positive integers: $F\colon n \rightarrow x_n$, $n = 1, 2, 3, \ldots$, or

$$F(n) = x_n. \tag{27}$$

We denote the operation of translation to the left by T; T is the operator which maps a sequence $x_1, x_2, x_3, \ldots$ into its translate $x_2, x_3, x_4, \ldots$, i.e., $TF = G$ is defined as

$$G(n) = F(n+1), \qquad n = 1, 2, 3, \ldots. \tag{28}$$

T is a linear operator: for any pair of sequences F_1, F_2 and constants α_1, α_2,

$$T(\alpha_1 F_1 + \alpha_2 F_2) = \alpha_1 T F_1 + \alpha_2 T F_2 .$$

A recursion relation of type (6), $x_{n+2} - a x_n - b x_{n+1} = 0$, can be expressed in terms of T by

$$T^2 F - bTF - aF = (T^2 - bT - aI)F = 0, \tag{29}$$

where I is the identity operator. (I maps every sequence into itself.)

Denote by S the set of all solutions of (29). S is the set of all sequences mapped into 0 by the linear operator

$$L = T^2 - bT - aI, \tag{30}$$

so S is a *linear space.* The first two terms $x_1 = F(1)$, $x_2 = F(2)$ may be prescribed arbitrarily; they and the 3 term recursion relation (6) determine all subsequent terms of a solution sequence uniquely. Thus the linear mapping $F \rightarrow (F(1), F(2))$ establishes a one-to-one correspondence between S and the two-dimensional space of all ordered pairs $(F(1), F(2))$. The space S is two-dimensional.

Finally, S is translation invariant; that is, T maps solutions of (29) into solutions of (29). This is a consequence of a basic theorem of linear algebra which states: If two operators commute, the null space of one is an invariant subspace of the other. In our case, let L be the operator (30); an element is in its null space if $LF = 0$, i.e., if F solves (29). L commutes with T since $LT = T^3 - bT^2 - aT = TL$. Hence S is an invariant subspace of T: a translate of a solution of (29) is a solution of (29).

Linear algebra teaches us that the action of a transformation T on a vector space S is best understood if eigenvectors of T are introduced as a basis in S.

Accordingly, we look for elements of S that are mapped by T into multiples of themselves; i.e., for sequences E for which $TE = \lambda E$. Let $E = \{x_1, x_2, \dots\}$; then $TE = \{x_2, x_3, \dots\} = \{\lambda x_1, \lambda x_2, \dots\}$. Thus

$$x_2 = \lambda x_1, \; x_3 = \lambda x_2 = \lambda^2 x_1, \; \dots \; x_n = \lambda^{n-1} x_1.$$

The recursion relation requires that

$$x_{n+2} - bx_{n+1} - ax_n = 0, \text{ i.e., } \lambda^{n+1}x_1 - b\lambda^n x_1 - a\lambda^{n-1}x_1 = 0$$

whence

$$\lambda^{n-1}x_1(\lambda^2 - b\lambda - a) = 0.$$

The first factor vanishes only for the sequence $(0, 0, \cdots)$ which is indeed a solution of (29). The other factor vanishes for the roots λ_1 and λ_2 of $\lambda^2 - b\lambda - a = 0$. This equation appeared on p. 6 as the characteristic equation (12′) of the matrix T defined there. [If we had chosen the solution sequences F_1, F_2 generated by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

respectively, as basis for the space S, then we could have expressed the translation operator as the 2×2 matrix $T = \left(\begin{smallmatrix} 0 & 1 \\ a & b \end{smallmatrix}\right)$ associated with (6) on p. 5].

If $\lambda_1 \neq \lambda_2$, there are two linearly independent eigen-sequences in S; they can be normalized so that

$$E^{(1)} = \{1, \lambda_1, \lambda_1^2, \dots \lambda_1^n, \dots\}, \; E^{(2)} = \{1, \lambda_2, \lambda_2^2, \dots, \lambda_2^n, \dots\},$$

and every solution F of (29) can be written in the form

$$F = c_1E^{(1)} + c_2E^{(2)}.$$

We turn now to differential equations. Let $F: x \to F(x)$ be an infinitely often differentiable function defined for real $x \geq 0$.

We denote the differentiation operator by D:

$$DF = F'. \tag{28′}$$

D is a linear operator: for any pair of functions F_1, F_2 and constants α_1, α_2

$$D(\alpha_1F_1 + \alpha_2F_2) = \alpha_1F_1' + \alpha_2F_2'.$$

We consider the differential equation

$$F'' - bF' - aF = (D^2 - bD - aI)F = LF = 0 \tag{29′}$$

where L abbreviates $D^2 - bD - aI$, and I is the identity operator (it maps every function into itself).

Denote by S the set of all solutions of (29′); in other words, let S be the null set of L. Then S is a linear space. According to a classical theorem of ordinary differential equations, a pair of arbitrarily prescribed initial values $F(0)$ and $F'(0)$ determines uniquely a solution of (29′). The correspondence of solutions F to their initial values $(F(0), F'(0))$ is an isomorphism of S and the two-dimensional vector space of initial values; so S is two-dimensional.

The operators D and L commute, so S is an invariant subspace of D: derivatives of solutions of (29′) are solutions of (29′).

Which elements of S are mapped, by D, into multiples of themselves? Those functions $E(x)$ for which $DE = \lambda E$, or $E'(x) = \lambda E(x)$.
This is true for exponential functions

$$E(x) = ke^{\lambda x}.$$

The differential equation (29′) requires that

$$E'' - bE' - aE = ke^{\lambda x}(\lambda^2 - b\lambda - a) = 0,$$

and this holds when $k = 0$, $E(x) \equiv 0$, and when λ is a zero of $\lambda^2 - b\lambda - a$, the same quadratic polynomial we met before. If $\lambda_1 \neq \lambda_2$, the linearly independent eigenfunctions in S are

$$E^{(1)}(x) = e^{\lambda_1 x}, \qquad E^{(2)}(x) = e^{\lambda_2 x},$$

and every solution of (29′) can be written in the form

$$F(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}.$$

There are countless generalizations of these principles, leading into functional analysis and its application to partial differential equations and related fields.

Purpose

The *Two-Year College Mathematics Journal* has been created to provide a communication forum for mathematicians interested and involved in the curricular and pedagogical problems of two-year colleges. The majority of such people are teaching at two-year colleges. They will contribute most of the articles. It is our hope, however, that all mathematicians interested in such problems, whether they teach at secondary, two- or four-year institutions, will use the Journal to communicate their ideas.

Editorial Policy

The Journal's policy is to publish articles that relate to curriculum and teaching as outlined in the Table of Contents. This Table of Contents has been designed specifically to encourage the broadest possible participation and representation. The editorial board is solely responsible for the selection of articles.