A Note on Angle Construction

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A numerical treatment of angle constructions is an excellent aid in various geometric procedures. This numerical approach goes well beyond the standard constructions of 30°, 45°, and 60° angles along with bisectors, perpendiculars, and certain regular polygons.

Many interesting facts in angle construction stem from a study of algebraic equations. An ideal place of beginning involves a fundamental theorem from the theory of equations. This theorem reads, "*If a cubic equation with integral coefficients has no rational roots, then none of its roots can be constructed with the Euclidean instruments*." Proofs of this theorem appear in various textbooks. Notable among these references is the work of W. V. Lovitt entitled *Elementary Theory of Equations* (chapter 14).

Building on this fundamental theorem, the trigonometric identity $\cos 3A = 4\cos^3 A - 3\cos A$ leads to the conclusion that a 20° angle cannot be constructed. Letting $A = 20°$, the equation becomes $\cos 60° = 4\cos^3 20° - 3\cos 20°$. If $\cos 20°$ is replaced by x, the equation $\frac{1}{2} = 4x^3 - 3x$ or its equivalent $8x^3 - 6x - 1 = 0$ results. This equation has no rational roots. Therefore x, which is $\cos 20°$, cannot be constructed. For if a 20° angle is constructible, then a right triangle is constructible whose acute angles measure 20° and 70°. If this right triangle has a unit hypotenuse, the side adjacent to the 20° angle would measure $\cos 20°$ (a contradiction as $\cos 20°$ would thus be constructible). More significantly, as a 60° angle can be constructed yet a 20° angle cannot, it follows that a general Euclidean method of angle trisection is impossible.

An instructive activity is to use the above result to decide the possibility of constructing angles having integral degree measure. Realizing that angle duplication, triplication, etc., are valid constructions, it can be asserted that a 1 degree angle is not constructible. For if so, by repetition, a 20 degree angle could be formed. A similar discussion follows for a 2 degree angle. It is interesting that the degree, which is the unit of sexagesimal measure, is not constructible. An analogous procedure shows that the smaller units called the minute and the second are neither constructible.

The possibility of constructing a 3 degree angle is established by first considering the regular pentagon construction. This regular pentagon construction also appears in the Lovitt reference mentioned above (chapter 14). Subtracting the 72 degree angle thus formed from a 75 degree angle (obtained from the summing of the constructible 30 degree angle and 45 degree angle) gives the desired 3 degree angle.

If an angle has integral degree measure, this measure will be of the form $3n$ or $3n + 1$ or $3n + 2$. If of the form $3n$, the construction is clearly possible by duplication, triplication, etc., of the 3 degree angle. The two other forms represent impossible constructions. If an angle whose measure is of the form $3n + 1$ could be constructed, then a 1 degree angle could likewise be constructed (by subtracting the angle measuring $3n$ degrees). Of course, the 1 degree angle construction was shown impossible. A similar discussion follows for angle measures of the form $3n + 2$.

These conclusions give a partial answer to the question of regular polygon constructibility. The following illustrate this point:

Example 1. The regular pentadecagon (15 sides) is constructible. This follows as the central angle has integral degree measure of the form $3n$, that is, 24°.

Example 2. The regular nonagon (9 sides) is not constructible. This follows as the central angle has integral degree measure not of the form $3n$, that is, 40°.

More generally, a regular polygon of integral central angle measure is constructible if and only if that measure is divisible by 3. As the interior angle is always the supplement of the central angle, its measure must also be divisible by 3.

The case for non-integral but rational measures is also of interest. To illustrate, a $22\frac{1}{2}$ degree angle can be constructed (by bisecting a 45 degree angle), but a $23\frac{1}{2}$ degree angle cannot be constructed at all. This latter angle would imply, by duplication, the construction of a 47 degree angle. This we have seen is impossible.

Next consider an angle whose measure is the positive rational number p/q. Moreover, let p and q be positive integers (the fraction p/q may be improper). It follows quickly that the angle construction is possible only if the numerator p is divisible by 3. Otherwise, multiplication by q (a valid construction) would result in an angle with integral measure p not divisible by 3. For example, one could say immediately that the angle measuring 4/7 of a degree is not constructible.

If the numerator is a multiple of 3, the angle may or may not be constructible. Clearly the angle measuring 3/2 degrees can be formed by bisecting a 3 degree angle. Various other angle sizes follow by repeated bisecting (3/4, 3/8, 3/16, etc.). Consider the possibility of constructing the angle measuring 3/7 of a degree. To further pursue this problem, use will be made of the result that a regular polygon having a prime number of sides p can be constructed if and only if p is of the form $2^{2^n} + 1$. This remarkable result also appears in the literature; an excellent treatment is given in the work of Louis Weisner entitled *Introduction to the Theory of Equations* (chapter 9, pp. 165–172). Numbers of the form $2^{2^n} + 1$ are called Fermat numbers. If n assumes the values 0, 1, 2, 3, and 4, then $2^{2^n} + 1$ assumes the respective prime values of 3, 5, 17, 257, and 65537. Hence the regular polygons of 3 sides, 5 sides, 17 sides, 257 sides, and 65537 sides are all constructible. There may be no primes of the form $2^{2^n} + 1$ beyond $n = 4$; this is presently a standing problem in mathematics. On the other hand, a regular seven-sided polygon cannot be constructed as 7 is a prime not of the form $2^{2^n} + 1$. Suppose now that an angle measuring 3/7 of a degree can be constructed. As an angle measuring 51 degrees can be formed (note that 51 is divisible by 3), an angle measuring $51\frac{3}{7}$ degrees would thus be constructible. This angle is precisely the central angle of a regular heptagon (seven-sided polygon). Since this polygon construction is impossible, neither can an angle measuring $3/7$ of a degree be constructed. It also follows from the above discussion that there is no general Euclidean method for dividing an angle into seven congruent parts.

A crucial question is "For what positive integers r is the angle measuring $3/r$ degrees constructible?". A well-known result (see Weisner, pp. 165–167) assists at this point: a regular polygon of n sides is constructible if and only if n is of the form $2^k (k > 1)$ or of the form $2^m (p_1)(p_2)(p_3) \cdots (p_t)$ where m is a non-negative integer and the p_i's are *distinct* Fermat primes.[1] This last mentioned product will be called a Fermat product.

> **Note:** A concise way of describing this condition is to say a regular n-sided polygon ($n > 2$) is constructible if and only if $\phi(n)$ is a power of 2. In this remark, $\phi(n)$ is the symbol for the familiar Euler ϕ-function (see the preceding reference).

Clearly, the angle measuring $3/r$ degrees is constructible if r is a power of 2. More generally, consider any positive integer r and let $3/r = 360/s$. In this form, it is evident that $3/r$ is constructible if and only if s is a Fermat product. For if s is of this form, $360/s$ is constructible as previously noted; this stems from the constructing of the regular s-sided polygon. If s is not of this form the construction is impossible. Otherwise a regular polygon of s sides could be constructed where s is not a Fermat product. Rather simply, the angle of $3/r$ degrees is constructible if and only if $120r$ is a Fermat product.

This matter of constructing the angle of $3/r$ degrees is far-reaching in consequence. It permits disposing of the rational constructibility problem entirely. If the angle measure reduces to the form $3/r$, the problem is solved as shown above. If not, the following considerations remain:

Case 1. Suppose the angle measure reduces to the form x/y where x is not a multiple of 3. This would imply by repetition the construction of an angle of x degrees. Of course, this is impossible as x is an integer not divisible by 3.

Case 2. Suppose the angle measure reduces to a form where the numerator is a multiple of 3. Represent this measure by the fraction $3n/r$ where n and r are relatively prime (there is no loss of generality in this restriction). We will show that $3n/r$ is constructible if and only if $3/r$ is constructible.

Proof.

1. Suppose $(n, r) = 1$; further suppose that the angle of $3n/r$ degrees is constructible.
2. Then $(3n, 3r) = 3$.
3. Expressing the greatest common divisor as a linear combination of $3n$ and $3r$, it follows that $a(3n) + b(3r) = 3$.
4. Now, $a(3n/r) + b(3r/r) = 3/r$. Clearly, each member of the left side of the equation represents a constructible angle. If either a or b is negative, an angle subtraction is implied (a valid construction).

[1] If the only Fermat primes are the five shown above, it can be stated that there are just 31 constructible regular polygons having an odd number of sides. This follows as ${}_5C_1 + {}_5C_2 + {}_5C_3 + {}_5C_4 + {}_5C_5 = 31$.

5. Thus, if $3n/r$ is constructible, then $3/r$ is constructible. By contraposition, if $3/r$ is not constructible, then $3n/r$ is not constructible.
6. The converse of step 5 is obvious; if $3/r$ is constructible, then $n(3/r)$ is constructible by repetition.

The foregoing discussion can be expressed rather concisely. An angle of rational degree measure is constructible if and only if that measure can be written in the form $3n/r$ where $(n, r) = 1$ and $3/r$ is constructible. Again, $3/r$ is constructible if and only if $120r$ is a Fermat product.

An illustration reinforces the rule for the rational case. Consider the problem of constructing an angle measuring $8\frac{2}{5}$ degrees. One could go through the linear combination procedure, thus writing $3/5$ as $45/5 - 42/5$. If $42/5$ is constructible, then $3/5$ is also (or by contraposition, if $3/5$ degrees is not constructible, then $42/5$ degrees is not constructible). As $3/5 = 360/600$, and $600 = (2^3)(3)(5^2)$, which is not a Fermat product, it follows that the angle of $3/5$ degrees cannot be constructed. Neither then can the angle of $8\frac{2}{5}$ degrees be constructed. Taking the alternate but more concise approach $8\frac{2}{5} = 42/5 = (14)(3/5)$. As 14 and 5 are relatively prime, and the angle of $3/5$ degrees is not constructible, it is thus established that the angle of $8\frac{2}{5}$ degrees cannot be constructed.

An abundance of fascinating sidelights stem from the above discussion. The one now given concerning the Morley triangle is typical.

The trisectors of the angles of any triangle intersect in points which are the vertices of an equilateral triangle (see Figure 1). Such an equilateral triangle is called the Morley triangle of the given triangle (Frank Morley, 1860–1937). An excellent article on this subject appeared in the September 1970 issue of the MATHEMATICS MAGAZINE; it is listed among the references. This triangle is a very impressive figure, rarely talked about in elementary mathematics. As angles in general cannot be trisected by the use of the unmarked straightedge and compass, a serious question arises, namely, "Can any constructible examples of the Morley triangle be found?". The answer to this question is "yes". In the discussion that follows, consideration will be given only to those triangles whose angles have integral degree measure. The reader may wish to pursue the matter more thoroughly (say through the rational case).

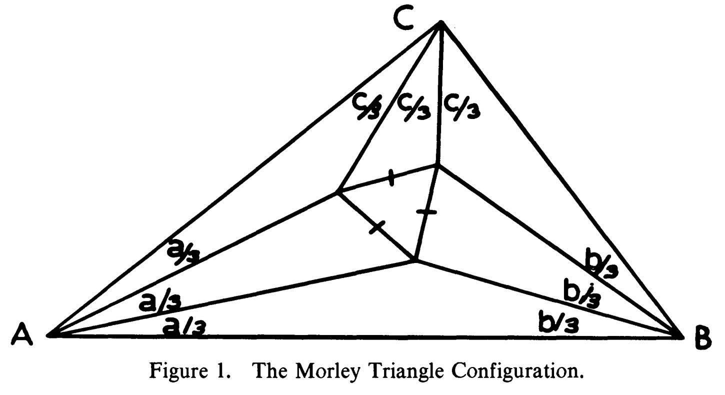

Figure 1. The Morley Triangle Configuration.

First, a theorem: *A constructible angle of integral degree measure can be trisected if and only if that integer is divisible by 9.* If 9 is a divisor of the angle's degree measure (call this measure $9n$), the trisection follows as one-third of $9n$ is equal to $3n$. As mentioned earlier, an angle whose degree measure is of the form $3n$ can be constructed. If the degree measure of the constructible angle is not divisible by 9, it will be either of the form $9n + 3$ or $9n + 6$. Proceeding by cases, one-third of $9n + 3$ is equal to $3n + 1$. The constructibility of $3n + 1$ would imply that of 1 degree, which (as shown earlier) is not possible. In like manner, the other case is rejected.

Consider now the constructible cases of angle trisectors, where the angles of the given triangle have integral degree measure. Let the measures of the angles A, B, and C be respectively $9x_1$, $9x_2$, and $9x_3$. As $9x_1 + 9x_2 + 9x_3 = 180$, then $x_1 + x_2 + x_3 = 20$. This last equation has various solutions in positive integers. For example, if $x_1 = 6$ and $x_2 = 6$, then $x_3 = 8$. The angles of the resulting triangle measure 54°, 54°, and 72°. Clearly each of these may be trisected as the trisections are respectively 18°, 18°, and 24° (multiples of 3°). It is fairly simple to list all cases of triangles based on positive integers satisfying $x_1 + x_2 + x_3 = 20$, thus having a more thorough list of constructible Morley triangles. This list will include the handiest of examples of a triangle whose Morley triangle is constructible, namely the right isosceles triangle.

It should be noted that the non-constructibility of the trisectors of an angle does not imply the non-existence of those trisectors. In other words, every triangle has a Morley triangle.

Should the Morley triangle be *constructible* for any triangle whatever, a general method of angle trisection would exist. This, as previously shown, is not the case. For example, it is not possible to construct the Morley triangle of an equilateral triangle. If so, the 60° angles of the equilateral triangle could thus be trisected, producing the non-constructible 20° angle in the process.[2]

Arguments thus far in the over-all discussion have centered around angles of rational degree measure. The case for angles of irrational degree measure is more difficult. In the concluding part of this note on angle constructions, only one part of the irrational case will be considered, namely the radian.

Clearly, the angle measuring one radian has an irrational degree measure; this familiar measure can be represented by $180/\pi$ degrees. As stated earlier, the basic unit of sexagesimal measure, the degree, is not constructible. A similar conclusion follows (surprisingly perhaps) for the basic unit of radian measure. The proof for non-constructibility involves a theorem of Lindemann's (C. Ferdinand Lindemann, 1852–1939). This theorem, appearing in the listed reference *Famous Problems and Other Monographs*, states that $x = \sin y$ (or equivalently, $2ix = e^{iy} - e^{-iy}$) has no solution in which x and y are both algebraic, except $x = 0$, $y = 0$. These angle measures are in radians. Note: a number is algebraic if and only if it can occur as a

[2] The side s of the Morley triangle of the triangle ABC is given by $s = 8(R)(\sin A/3)(\sin B/3)(\sin C/3)$, where R is the circumradius of triangle ABC. It can be shown that for a 30°–60°–90° triangle and $R = 1$ that $s = 2(\cos 10°) - \sqrt{3}$ (a non-constructible length).

root of an algebraic equation; the number is transcendental otherwise. Moreover, all transcendental lengths are non-constructible (Weisner, pp. 159–161); this includes the number pi (π). As the number 1 is algebraic, it follows that $x = \sin 1$ is not algebraic, and hence not constructible. As sin 1 is not constructible, neither is 1 radian constructible.

Since every rational multiple of 1 is algebraic, it follows by the preceding theorem that no rational multiple of a radian can be constructed (except of course the zero multiple). The theorem is far-reaching; it establishes even that no algebraic multiple of a radian (except zero) is constructible.

If the mil were equal to exactly one-thousandth of a radian, its non-constructibility would immediately follow. Actually, this is an approximation. The mil is defined to be $1/6400$ of a revolution. In degrees, this measure is exactly $360/6400$. As 6400 is equal to $(2^8)(5^2)$, neither is the mil constructible. Note that the factorization above is not a Fermat product.

A number of interesting observations stem from the consideration of the possibility of constructing one radian. If this construction were possible, the associated sector of the unit circle would have an area of $1/2$. Such a sector could thus be squared, i.e., a square equal in area to this sector could be constructed. Though this is hardly the "squaring of a circle," it is nevertheless an impressive consequence of assuming radian construction. Such a remark raises the question (not pursued in this article) "for what constructible central angle measures can the associated sector of a circle be squared?".

Note: Lindemann's theorem establishes the transcendence of π. Since $1 = \sin(\pi/2)$ and 1 is algebraic, it follows that $\pi/2$ (and hence π) is transcendental. Should a method exist for squaring a circle, then the unit circle with area $\pi(1)^2 = \pi$ would give rise to the square with side measuring $\sqrt{\pi}$. As squaring a line segment length is a valid construction, π would thus be constructible. The known impossibility of constructing π hence establishes that squaring a circle is impossible.

The preceding account of constructions calls attention to the surprising impossibility of constructing (with Euclidean tools) the conventional units of angle measure (the degree, minute, and second, along with the radian and the mil). One may naturally wonder about the desirability of using a non-constructible unit. The degree choice is of interesting motivation and can be pursued in the various accounts of the history of mathematics. Likewise the radian choice has computational and simplifying value far outweighing the non-constructible feature.

Additional angle measure types could be included. Hopefully the few above prove adequate to point out the importance of this supplementary, numerical approach to angle constructions.

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