The Discovery of a Generalization: An Example in Problem Solving

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Since Descartes' development of the rectangular coordinate system, mathematicians have used the following analytic geometry formula for finding the squared distance between two points, $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$,

$$\overline{P_1P_2}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2. \tag{1}$$

Unfortunately, many students do not consider this formula to be of much interest and rarely do they appreciate its power as a mathematical tool. One reason for this lack of interest and appreciation of the distance formula is that students are too often asked to merely prove the formula or to verify that the formula does in fact yield the desired distances for a set of exercises. Although such a task does provide practice in translating from geometric form to algebraic form, together with practice in algebraic manipulation, it ignores the creative aspect of mathematics.

This article will outline a problem solving approach to a topic in geometry that was original research for us (the results may be known to other mathematicians). The process described below affords students ample opportunity for transforming to algebraic forms and for algebraic manipulation. In addition, it indicates how a student might develop a conjecture and then verify it for himself.

Recall that for two equally spaced points V_1 and V_2 on a unit circle and for any other point P on the circle that V_1V_2 is a diameter and ΔV_1PV_2 is a right triangle (see Figure 1). From the Pythagorean theorem we have

$$\overline{PV_1}^2 + \overline{PV_2}^2 = \overline{V_1V_2}^2 = 4.$$

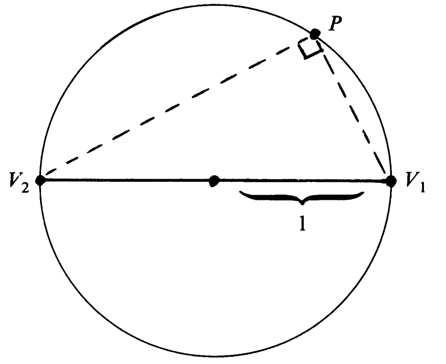

Figure 1.

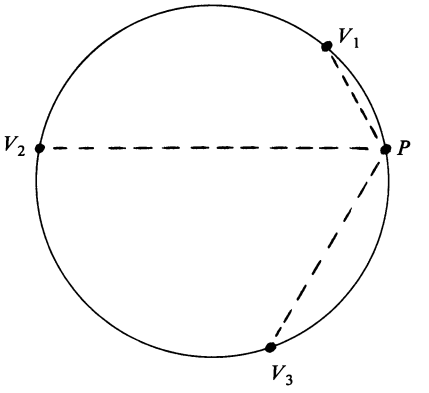

Figure 2.

Our first question is: Can this result be generalized?

Suppose we take V_1, V_2 and V_3 to be three equally spaced points on a unit circle and let P be any point on the circle (see Figure 2). The sum of the squares of $\overline{PV_1}$, $\overline{PV_2}$, $\overline{PV_3}$ becomes $\overline{PV_1}^2 + \overline{PV_2}^2 + \overline{PV_3}^2 \doteq 6.000$ (see Table 1). The data in Tables 1 and 2 was obtained from the computer program listed at the end of this article. (At the beginning of our research the data was obtained by making scale drawings, and then measuring and performing the necessary computations).

Table 1

$\overline{PV_1}^2 + \overline{PV_2}^2 + \overline{PV_3}^2 + \cdots + \overline{PV_n}^2.$

Arbitrarily selected points	Number of points (n) on the unit circle				
	2 points	3 points	4 points	5 points	6 points
P_1	*4.000	6.000	8.000	10.000	12.000
P_2	4.000	6.000	8.000	10.000	12.000
P_3	4.000	6.000	8.000	10.000	12.000
P_4	4.000	6.000	8.000	10.000	12.000
P_5	4.000	6.000	8.000	10.000	12.000
mean of sum	4.000	6.000	8.000	10.000	12.000

* Each value is rounded to nearest thousandth.

From Table 1 we observe,

$$\overline{PV_1}^2 + \overline{PV_2}^2 + \overline{PV_3}^2 + \overline{PV_4}^2 \doteq 8 = 2(4) \text{ (for 4 points)},$$

$$\overline{PV_1}^2 + \overline{PV_2}^2 + \overline{PV_3}^2 + \overline{PV_4}^2 + \overline{PV_5}^2 \doteq 10 = 2(5) \text{ (for 5 points)},$$

$$\overline{PV_1}^2 + \overline{PV_2}^2 + \overline{PV_3}^2 + \cdots + \overline{PV_6}^2 \doteq 12 = 2(6) \text{ (for 6 points)}.$$

Thus, **conjecture 1** is

For n ($n \geqslant 2$) equally spaced points ($V_1, V_2, \ldots, V_n$) on a unit circle and P any point on the circle, the sum

$$\overline{PV_1}^2 + \overline{PV_2}^2 + \overline{PV_3}^3 + \cdots + \overline{PV_n}^2 = 2n. \tag{2}$$

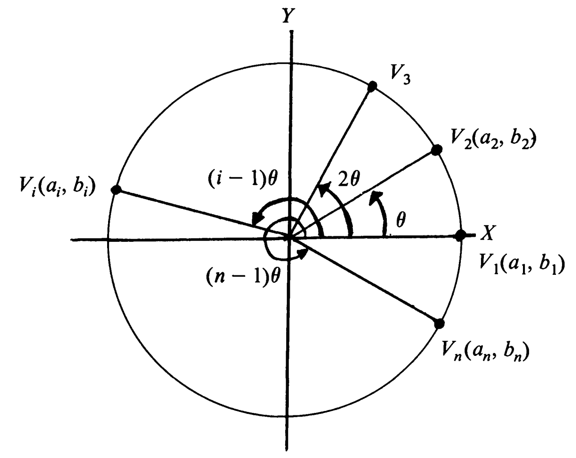

Figure 3.

Proof. A graphic representation is shown in Figure 3. To simplify the algebraic manipulation and with no loss of generality, we will restrict the point V_1 to be the intersection of the circle and the positive X axis. Let the coordinates of V_i and P be (a_i, b_i) and (x, y), respectively. From the distance formula we have $\overline{PV_i}^2 = (x - a_i)^2 + (y - b_i)^2$ and thus the desired sum S becomes

$$\begin{aligned} S &= \overline{PV_1}^2 + \overline{PV_2}^2 + \overline{PV_3}^2 + \cdots + \overline{PV_n}^2 \\ &= \left[(x-a_1)^2 + (y-b_1)^2\right] + \left[(x-a_2)^2 + (y-b_2)^2\right] \\ &\quad + \cdots + \left[(x-a_n)^2 + (y-b_n)^2\right] \\ &= \left[(x^2 - 2xa_1 + a_1^2) + (y^2 - 2yb_1 + b_1^2)\right] + \left[(x^2 - 2xa_2 + a_2^2)\right. \\ &\quad \left. + (y^2 - 2yb_2 + b_2^2)\right] + \cdots + \left[(x^2 - 2xa_n + a_n^2) + (y^2 - 2yb_n + b_n^2)\right]. \end{aligned}$$

Simplifying and using summation notation (where the sum is from $i = 1$ to n) we have,

$$S = n(x^2 + y^2) - (2x)\sum a_i - (2y)\sum b_i + \sum (a_i^2 + b_i^2).$$

Since the coordinates of P and V_i satisfy the equation of the circle ($X^2 + Y^2 = 1$), we have $x^2 + y^2 = 1$ and $a_i^2 + b_i^2 = 1$. So,

$$S = n(1) - (2x)\sum a_i - (2y)\sum bi + \sum(1)$$

$$= n - (2x)\sum a_i - (2y)\sum b_i + n$$

$$= 2n - (2x)\sum a_i - (2y)\sum b_i.$$

The desired result will be obtained if $\sum a_i = 0$ and $\sum b_i = 0$. To prove these sums are zero, we will use two different arguments.*

Think of the n points as equal weights distributed on the outer edge of a homogeneous disc. Since the disc will balance on its center, the x and y moments must each add to zero. Thus, $\sum a_i = \sum b_i = 0$.

A more formal proof will now be given. Let the ith point (V_i) on the unit circle be represented by the complex number $Z_i = a_i + b_i\sqrt{-1}$. Since the n points ($V_1, V_2, \ldots, V_n$) are equally spaced on a unit circle, we have the Z_i's are the nth roots of unity. Thus, they are the n roots of the polynomial equation $Z^n - 1 = 0$. So, $(Z - Z_1)(Z - Z_2) \cdots (Z - Z_n) = Z^n - 1$. But the coefficient of Z^{n-1} is $\sum Z_i$. Hence, $\sum Z_i = 0$. Therefore, both the real part $\sum a_i$ and the imaginary part $\sum b_i$ must equal zero. Now combining these results, we have,

$$S = \overline{PV_1}^2 + \overline{PV_2}^2 + \overline{PV_3}^2 + \cdots + \overline{PV_n}^2 = (2)(n) \quad \text{for } n \geqslant 2.$$

Thus, Conjecture 1 has been verified. Recall the circle in Conjecture 1 was a unit circle. A second question that might be asked is: How is Conjecture 1 changed (if any) if the radius r of the circle is permitted to vary?

The data in Table 2 indicates a pattern that allows us to suggest a plausible answer to the question, which we designated as **Conjecture 2.**

For any point P on a circle of radius r and n ($n \geqslant 2$) equally spaced points on the circle, the sum

$$\overline{PV_1}^2 + \overline{PV_2}^2 + \overline{PV_3}^2 + \cdots + \overline{PV_n}^2 = r^2(2n). \tag{3}$$

The proof is similar to the proof of Conjecture 1 and thus will not be given.

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* We are indebted to an unknown referee for both of these proofs. Our original proof of $\sum a_i = \sum b_i = 0$ made use of two trigonometric series.

Table 2

$\overline{PV_1}^2 + \overline{PV_2}^2 + \cdots + \overline{PV_n}^2.$

Number of Points	Radius (R) of Circle				
(n)	$R = 1$	$R = 2$	$R = 3$	$R = 4$	$R = 5$
2	*4.000	$16 = 4 \cdot (4)$	$36 = 9 \cdot (4)$	$64 = 16 \cdot (4)$	$100 = 25 \cdot (4)$
3	6	$24 = 4 \cdot (6)$	$54 = 9 \cdot (6)$	$96 = 16 \cdot (6)$	$150 = 25 \cdot (6)$
4	8	$32 = 4 \cdot (8)$	$72 = 9 \cdot (8)$	$128 = 16 \cdot (8)$	$200 = 25 \cdot (8)$
5	10	$40 = 4 \cdot (10)$	$90 = 9 \cdot (10)$	$160 = 16 \cdot (10)$	$250 = 25 \cdot (10)$
6	12	$48 = 4 \cdot (12)$	$108 = 9 \cdot (12)$	$192 = 16 \cdot (12)$	$300 = 25 \cdot (12)$

* Each value is the mean of the sum (rounded to the nearest thousandth) for ten arbitrarily chosen points P.

In Conjectures 1 and 2 the variable point P has always been taken to be on the circle. This suggests a third question: What will happen if the point P is any point in the plane of the circle?

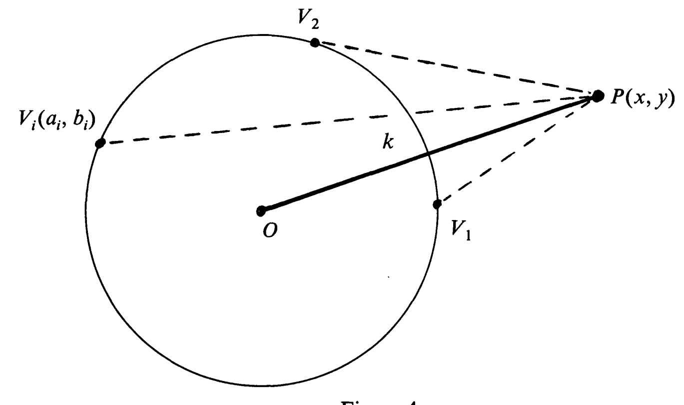

Figure 4.

To answer this question we will proceed directly to an analytic approach to the problem. Figure 4 illustrates the situation. Let $\overline{PO} = k$. Then,

$$
\begin{aligned}
S &= \overline{PV_1}^2 + \overline{PV_2}^2 + \cdots + \overline{PV_n}^2 \\
&= \left[(x - a_1)^2 + (y - b_1)^2\right] + \left[(x - a_2)^2 + (y - b_2)^2\right] \\
&\quad + \cdots + \left[(x - a_n)^2 + (y - b_n)^2\right] \\
&= \left[\left(x^2 - 2xa_1 + a_1^2\right) + \left(y^2 - 2yb_1 + b_1^2\right)\right] + \cdots \\
&\quad + \left[\left(x^2 - 2xa_n + a_n^2\right) + \left(y^2 - 2yb_n + b_n^2\right)\right].
\end{aligned}
$$

Simplifying and using summation notation we have

$$S = n(x^2 + y^2) - (2x)\sum a_i - (2y)\sum b_i + \sum (a_i^2 + b_i^2)$$

$$= n(x^2 + y^2) - 0 - 0 + nr^2$$

because

$$\sum a_i = \sum b_i = 0 \quad \text{and} \quad a_i^2 + b_i^2 = r^2.$$

Since $\overline{PO} = k$, we have $(x - 0)^2 + (y - 0)^2 = k^2$. Thus,

$$S = n(k^2) + nr^2 = n(k^2 + r^2).$$

Hence, we have proven the following theorem:

For $n (n \geqslant 2)$ equally spaced points $(v_1, v_2, \ldots, v_n)$ on a circle of radius r and center at 0 and P any point in the plane of the circle, the sum of the squares of the distances from P to v_i $(i = 1, 2, \ldots, n)$ is $n(k^2 + r^2)$, where $\overline{PO} = k$. This may be denoted by

$$\overline{PV_1}^2 + \overline{PV_2}^2 + \overline{PV_3}^2 + \cdots + \overline{PV_n}^2 = n(k^2 + r^2). \qquad (4)$$

Some properties of regular polygons may be obtained by using the results developed here. For example, let $A_1A_2A_3 \ldots A_n$ be a regular polygon (henceforth called a polygon) of n sides inscribed in a circle of radius r and let P be any point on the circle (see Figure 5 for $n = 6$). Since the vertices of the polygon are equally spaced

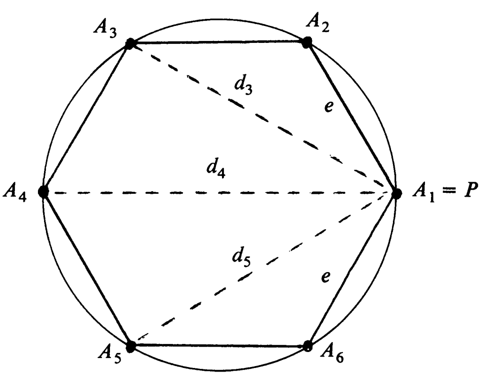

Figure 5.

and on the circumscribed circle, then formula 3 from conjecture 2 may be applied. Since P can be any point on the circle, we will take P to coincide with vertex A_1, i.e., $P = A_1$. Let $d_J = \overline{A_1A_J}$ (for $J \neq 1, 2$ or n) and e equal the length of a side of the polygon. Note that d_J defined above is the diagonal from vertex A_1 to A_J. So, we have

$$S_1 = \overline{PA_1}^2 + \overline{PA_2}^2 + \cdots + \overline{PA_J}^2 + \cdots + \overline{PA_{n-1}}^2 + \overline{PA_n}^2$$
$$= 0 + e^2 + d_3^2 + d_4^2 + \cdots + d_J^2 + \cdots + d_{n-1}^2 + e^2$$
$$= 2nr^2 \quad \text{(by formula 3).}$$

This is the sum of the square of each diagonal from vertex A_1 and two sides of the polygon. When a similar sum is obtained at each vertex, we have the sum of the squares of all diagonals and all sides of the polygon counted twice. Therefore, the sum of the square of each side and each diagonal of a regular polygon is $(n/2)(2nr^2)$ or $(nr)^2$.

We believe the ideas and techniques presented in this article are of value in a junior college curriculum concerned with the method of problem solving along with the accumulation of algebraic skills. By discovering, testing and verifying relationships, many students may experience for the first time the creative aspect of mathematics.

The following program in the BASIC language was used to obtain the data for Tables 1 and 2.

```
5 PRINT 'WHAT ARE THE VALUES OF THE RADIUS (R), THE
  NUMBER'
8 PRINT 'OF EQUALLY SPACED POINTS (N), AND THE X'
10 PRINT 'COORDINATE (FOR -R < = X < = R) OF THE POINT P';
20 INPUT R, N, X
30 LET Y = SQR (R**2 - X**2)
40 LET T = 2*3.1416/N
50 FOR J = 1 TO N
60 LET A = R* COS((J - 1)*T)
70 LET B = R* SIN ((J - 1)*T)
80 LET D2 = (X - A)**2 + (Y - B)**2
90 LET S = S + D2
100 NEXT J
110 LET S = INT (1000*S +.5)/1000
120 PRINT
130 PRINT 'FOR R ='; R; 'N ='; N; 'AND THE POINT P ('; X; ','; Y; ')'
140 PRINT 'THE SUM IS'; S
150 END
```

One winter day (back in the 1930's) a visiting Polish mathematician complained that his room was so hot that he couldn't sleep. I asked him, "Did you try opening the window?" He answered, in tones of shocked incredulity, "In Poland we don't open windows."

Ralph P. Boas