For I take it that a justification for a test item must be that it asks the sort of question which the student is likely to meet when he is doing mathematics. In 35 years of doing mathematics as a professional, I have never yet met a situation in which I knew that the answer was one of four possibilities and I simply had to make the correct choice. Moreover, were I ever placed in such a situation artificially,[1] it is highly unlikely that I would use as a method of solving the problem the method appropriate to a situation in which I had not been told that the correct answer figured in a set of four potential solutions.

I have given about six especially pernicious features of a traditional curriculum. I do not wish to weary the reader by prolonging the list, but I do want to say that the list is by no means complete. Let this list, however, suffice to establish my thesis that math avoidance is a natural reaction to so much of the experience of the student in situations euphemistically described as the learning of mathematics.

[1] The 'ink-blot' problem is an example of an artificial situation for a multiple choice question of which I entirely approve as a means of instilling mathematical thinking. An arithmetical problem and, say, four possible solutions are displayed with certain digits obscured by ink-blots. The student must determine, by elimination, which solution is correct.

In Part 2, Peter Hilton discusses the constituents of a sound mathematics education and deals with the current situation of mathophobic adults.

COMPUTERS & CALCULATORS

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In this section readers are encouraged to share their experiences with computers and calculators as they apply to the two-year college mathematics curriculum. There is special interest in innovative uses of these tools to solve problems, to present concepts, and to define new directions for curriculum development. All material for this section should be sent to Thomas Green. (See inside cover for details on submitting manuscripts.) Be sure to include with your paper a copy of the computer or calculator program and a successful run and output.

Calculators to Motivate Infinite Composition of Functions

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In recent years, a considerable amount of attention has been given to the use of computers and calculators in motivating mathematical concepts such as limits and infinite series. In this paper it will be shown how calculators may be used to motivate a concept called infinite composition of functions which will contain, as *special cases*, several mathematical topics, such as continued square roots, continued fractions, and infinite products. The concept of infinite composition of functions is rich enough to spawn creative thinking as well as to provide rewarding exercises.

Infinite Composition

Select any positive number x and press the square root key ($\sqrt{x}$) as often as you wish. The displayed numbers seem to approach 1. Now this brings up an interesting question. If you had an ideal machine, one that would compute with all the digits and display all the digits, and if you pressed the key "continually," would the display "approach" 1? Although this hypothesis was formed by observing the calculator, its proof lies in the proper mathematical formulation of the problem.

We may formulate the problem as follows. The first time we press the key we evaluate the function $f(x) = x^{1/2}$, the second time yields $f(f(x)) = x^{1/2^2}$, the third time yields $f(f(f(x))) = x^{1/2^3}$; therefore, the nth time we press the key we evaluate $x^{1/2^n}$. Pressing the key "continually" can then be modeled as letting n get large, *which results in* $x^{1/2^n}$ *approaching* x^0, *which equals 1 for* $x > 0$.

Hence, we have used mathematics to *prove* a hypothesis suggested by the calculator! But what is so special about $f(x) = \sqrt{x}$? Let us try to generalize the technique.

Consider a real valued function f having the property that its range is a subset of its domain. Let us define $S_1(x) = f(x)$, $S_2(x) = f(f(x))$, $S_3(x) = f(f(f(x)))$, and so on; for each integer n we have $S_n(x)$. We define the infinite composition of f at x, denoted $S_\infty(x)$, as the limit of $S_n(x)$ as n gets large (approaches infinity), provided the limit exists.

This formulation clearly covers the $\sqrt{x}$ example. We will now consider other examples along with calculator displays. For our calculations we have used an algebraic language calculator (Texas Instruments, SR-52), and it should be noted that different calculators may require different sequence steps.

Example 1. Let $f(x) = x^2$. Note each time we press the x^2 key we take the composition of f. Hence, can we "guess" the infinite composition of f at, say, $x = 0.5$, by observing the following calculator display sequence?

Step	Key	Sequence	Display
1	.5	x^2	$S_1(.5) = 0.25$
2		x^2	$S_2(.5) = 0.0625$
3		x^2	$S_3(.5) = 0.00390625$
4		etc.	$S_4(.5) = 0.0000152588$
5			$S_5(.5) = 0.0000000002$
⋮			⋮
∞			$S_\infty(.5) = ?$

Did you guess $S_\infty(.5) = 0$? Is this correct? For what other values of x would you get a similar pattern? Again, we must use the mathematics that we have developed before we can hope to get a satisfactory answer to these questions. In so doing, we have $S_1(x) = x^2$, $S_2(x) = x^{2^2}, \ldots, S_n(x) = x^{2^n}$, and now it is clear $S_\infty(x) = 0$ whenever x is between -1 and 1.

We should observe that, for the infinite composition problems that we have considered so far, the key to their mathematical proof has been in obtaining a "nice" expression for $S_n(x)$. For some rather interesting problems a neat expression for $S_n(x)$ may not be easily obtained, and hence we are now motivated to extend our theory to try to alleviate this problem. Very few things in life are free, and in this case there is no exception. We elect to pay the price of assuming f is continuous at $S_\infty(x)$. This allows us to take the limit "inside" the composition; that is,

$$S_\infty(x) = \operatorname*{limit}_{n\to\infty} S_n(x) = \operatorname*{limit}_{n\to\infty} S_{n+1}(x) = \operatorname*{limit}_{n\to\infty} f(S_n(x))$$

$$= f\Big(\operatorname*{limit}_{n\to\infty} S_n(x)\Big) \quad \text{(Did you see the limit go "inside" the function?)}$$

$$= f(S_\infty(x)).$$

Eureka! We have proved the following results.

Theorem. *If f is continuous at $S_\infty(x)$ for all x such that $S_\infty(x)$ exists, then $f(S_\infty(x)) = S_\infty(x)$.*

Corollary (Fixed point). *In addition to the assumptions of the Theorem, if we have $S_\infty(x) = k$, where k is a constant, then $f(k) = k$.*

Example 2 (Continued Square Roots). Let $f(x) = \sqrt{a + x}$ where $a > 0$ and $x \geqslant -a$. Let us try to "guess" the infinite composition of f using $a = 12$ and $x = 1$ by observing the following calculator display.

Step	**Key Sequence**	**Display**
1	[1] [+] [12] [=] [$\sqrt{x}$]	$S_1(1) = 3.605551275$
2	[+] [12] [=] [$\sqrt{x}$]	$S_2(1) = 3.950386219$
3	[+] [12] [=] [$\sqrt{x}$]	$S_3(1) = 3.993793462$
4	etc.	$S_4(1) = 3.999224108$
5		$S_5(1) = 3.999903012$
6		$S_6(1) = 3.999987877$
7		$S_7(1) = 3.999998485$
8		$S_8(1) = 3.999999811$
9		$S_9(1) = 3.999999976$
10		$S_{10}(1) = 3.999999997$
11		$S_{11}(1) = 4.$
⋮		⋮ Display remains unchanged?
∞		$S_\infty(1) = ?$

It appears the "logical guess" is $S_\infty(1) = 4$. Using the mathematical approach we have

$$S_1(x) = \sqrt{a + x}\ ,$$

$$S_2(x) = \sqrt{a + \sqrt{a + x}}\ ,$$

$$S_3(x) = \sqrt{a + \sqrt{a + \sqrt{a + x}}}\ ,$$

and hence

$$S_\infty(x) = \sqrt{a + \sqrt{a + \sqrt{a + \sqrt{a + \cdots}}}}$$

provided $S_\infty(x)$ exists. So what! How do we "evaluate" something in such a complicated form? Maybe we should appeal to our theorem; f is certainly continuous! Applying the theorem we have $f(S_\infty(x)) = S_\infty(x)$ and hence $\sqrt{a + S_\infty(x)} = S_\infty(x)$, which yields $S_\infty(x) = (1 + \sqrt{1 + 4a}\,)/2$ since $S_\infty(x) > 0$. Alas! For $a = 12$, $S_\infty(x) = 4$ for all $x > -12$, and certainly for $x = 1$ used in our calculator demonstration. In fact $f(4) = 4$, thus illustrating the fixed point corollary. This example of continued square roots was previously considered by Rotando (1965) and is presented here to show that it is simply another interesting example of an infinite composition of a function and that it contains a special application of our theorem.

Example 3 (Continued Fractions). Let $f(x) = b/(a + x)$ with $a > 0$, $b > 0$, and $x > 1$. Again, let us try to guess $S_\infty(2)$ using the calculator with $x = 2$ and $a = b = 1$. With a little endurance, we obtain the following.

Step	Key Sequence	Display
1	[2] [+] [1] [=] [$\frac{1}{x}$]	$S_1(2) = .3333333333$
2	[+] [1] [=] [$\frac{1}{x}$]	$S_2(2) = .75$
3	[+] [1] [=] [$\frac{1}{x}$]	$S_3(2) = .5714285714$
4	[+] [1] [=] [$\frac{1}{x}$]	$S_4(2) = .6363636364$
5	etc.	$S_5(2) = .6111111111$
⋮		⋮
23		$S_{23}(2) = .6180339885$
24		$S_{24}(2) = .6180339888$
25		$S_{25}(2) = .6180339887$
26		$S_{26}(2) = .6180339888$

27	$S_{27}(2) = .6180339887$
28	$S_{28}(2) = .6180339888$
29	$S_{29}(2) = .6180339887$
30	$S_{30}(2) = .6180339888$
31	$S_{31}(2) = .6180339888$
32	$S_{32}(2) = .6180339888$
⋮	⋮
∞	$S_{\infty}(2) = ?$

We now prove that, if you guessed $S_{\infty}(2) = .6180339888$, you are correct to nine decimal places, not ten!

We have

$$S_1(x) = \frac{b}{a+x}, \quad S_2(x) = \frac{b}{a + \frac{b}{a+x}}, \quad S_3(x) = \frac{b}{a + \frac{b}{a + \frac{b}{a+x}}},$$

and hence

$$S_{\infty}(x) = \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{a + \cdots}}}}.$$

Assuming $S_{\infty}(x)$ exists, we apply the theorem to obtain $f(S_{\infty}(x)) = S_{\infty}(x)$, which results in $b/(a + S_{\infty}(x)) = S_{\infty}(x)$. Using the fact that $S_{\infty}(x) > 0$, we obtain

$$S_{\infty}(x) = \frac{-a + \sqrt{a^2 + 4b}}{2}.$$

In fact, for $a = 1$ and $b = 1$, we have $S_{\infty}(x) = (-1 + \sqrt{5})/2 = .6180339887\ldots = G$, which is the well-known constant referred to as the "Ratio of the Golden Section."

Example 4 (S_{∞} does not exist). Let $f(x) = x^2$ for $x > 1$. Now $S_n(x) = x^{2^n}$ and $\lim_{n\to\infty} S_n(x) = \infty$ for each $x > 1$, and hence $S_{\infty}(x)$ does not exist. Select $x = 3$ and continually press the x^2 key. Observe the result!

Example 5 (S_{∞} does not exist). Let $f(x) = 1/x$ for $x \notin \{-1, 0, 1\}$. We have $S_1(x) = 1/x$, $S_2(x) = x$, $S_3(x) = 1/x$, and hence $S_n(x)$ equals x if n is even and $1/x$ if n is odd. Note, as $n \to \infty$, S_n oscillates between x and $1/x$, and hence $\lim_{n\to\infty} S_n(x)$ does not exist. Select $x = 3$ and continually press the $1/x$ key. Observe the result!

It is interesting to observe the difference in the manner in which the calculator reacts to Example 4 as compared with Example 5.

A Geometric Interpretation

An intuitive feeling may be obtained (see Figure 1) for the infinite composition of a function by considering the example $f(x)=\sqrt{x}$ where $x>1$. The composition is "closing down" on the unit box and thus approaching 1. Also notice the role the line $y=x$ is playing concerning the fixed point corollary.

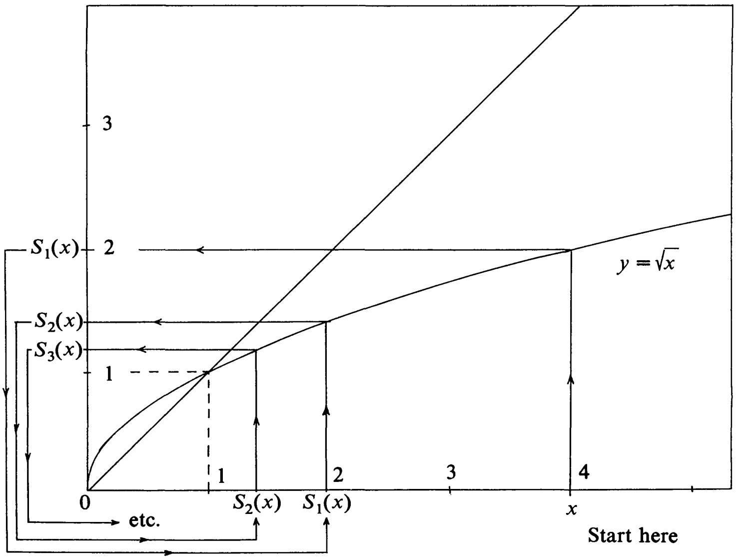

Figure 1.

Generalization

Instead of just one function, let us now consider a sequence of functions f_1, f_2, $f_3, \ldots$. We now have $S_1(x)=f_1(x)$, $S_2(x)=f_2(f_1(x))$, $S_3(x)=f_3(f_2(f_1(x)))$, and so on; for each integer n we have $S_n(x)$. We define the infinite composition of the sequence of functions of x, denoted $S_\infty(x)$, as the limit of $S_n(x)$ as n approaches ∞, provided the limit exists!

Example 6 (Infinite Products). Let $f_i(x)=xA_i$ for $i=1,2,\ldots$. We have $S_1(x)=xA_1$, $S_2(x)=xA_1A_2$, and hence $S_n(x)=x\prod_{i=1}^{n}A_i$. Therefore, $S_\infty(x)=x\prod_{i=1}^{\infty}A_i$, and hence the concept of infinite products becomes a special case of infinite composition of functions. An interesting exercise is to obtain a general expression for the A_i's and the value of x in order to obtain the well-known expansion

$$\pi = 4\,\frac{2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8\cdot10\cdot10\cdot12\cdot12\cdots}{3\cdot3\cdot5\cdot5\cdot7\cdot7\cdot9\cdot9\cdot11\cdot11\cdot13\cdots}$$

and to "investigate" this product on the calculator.

It should be noted that if we had selected $f_i(x)=x+A_i$ then it follows in a similar manner that infinite series become a special case of infinite composition of functions.

General Questions

The material available through infinite composition of functions is abundant. A few (out of many) questions that may be asked are as follows:

1. What infinite compositions can be obtained using a sequence of trigonometric functions? What about selecting a sequence of inverse trigonometric functions? Is the infinite composition of arctan(x) the constant function $S_\infty(x) = 0$ for all real numbers x? What about the infinite composition of functions, such as cos(x) and sin(x)?
2. What is the relationship between a sequence of functions and the sequence of inverse functions in terms of their convergence in infinite composition?
3. What properties must be satisfied in order for a sequence of functions to converge in infinite composition to a continuous function? To a differentiable function? To an integrable function?
4. It is well known that, in general, composition of functions is not commutative. Hence, what could be gained by considering the infinite composition of the functions in a different order, such as

$$S^*_\infty(x) = \operatorname*{limit}_{n\to\infty} f_1(f_2(\cdots f_n(x)\cdots)).$$

With this expansion, could a more general form of continued fractions be obtained? Would this particular form be compatible with the calculator?

Conclusions

A concept called infinite composition has been introduced which unifies many mathematical topics. The concept is revealed to be as elementary in nature as simply pressing keys on a calculator but yet solid enough in mathematical content to challenge the inquisitive mind of a student at *any level*. The calculator is considered as an interesting tool with which students can make observations and form intriguing hypotheses; however, it should be stressed that hypotheses found on the calculator cannot be proved on the calculator. Only with the proper mathematical formulation can the student hope to carry out his inquiry. Proofs often lead to interesting and useful mathematics. Therefore, the calculator plays an *important first step* in the process. In this vein, it is believed that the student will gain a better appreciation not only of the capabilities of the calculator but also of its limitations.

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