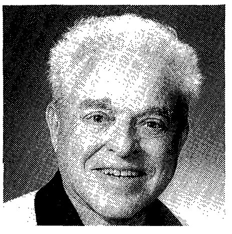
*Two Surprising Theorems on Cavalieri Congruence**

Howard Eves



Howard Eves received his mathematical education at Virginia, Harvard, and Princeton. His major interests are geometry, problem solving, and history. He has written over 25 books and published a large number of articles in mathematics, physics, engineering, and pedagogical journals, and in encyclopedias and other reference works. Among many editorial duties, he served for 25 years as a problems editor of *The American Mathematical Monthly*. He has lectured in almost every state of the Union, and is a recipient of many awards. Retired from the University of Maine, he is at present a Distinguished Visiting Professor at the University of Central Florida.

In the fourteenth century, the Blessed John Colombini of Siena founded a religious group known as the *Jesuats*, which was in no way related to the *Jesuits*. The order was approved by Pope Urban V in 1367. The original work of the order was the care of those stricken by the Black Death, which raged over Europe at the time, and the burial of the fatally smitten. With the passage of time the Jesuat order diminished, and in 1606 an attempt at a revival was made. But certain abuses later crept into the order, with the result that the group now no longer exists. It seems that the manufacture and sale of distilled liquors, apparently in a manner unacceptable by Canon Law, along with a growing scarcity of members, led to the order's suppression by Pope Clement IX in 1668.

In 1613, only a few years after the attempted revival of the Jesuats, a young fifteen-year-old Italian boy named Bonaventura Cavalieri was accepted as a member of the order, and then spent the rest of his life in its service. It is because of this, and because of the ultimate vanishing of the order and the natural confusion between Jesuat and Jesuit, that so many major encyclopedias, histories, and source books erroneously state that Cavalieri was a Jesuit, instead of a Jesuat, furnishing an excellent example of written histories containing a hidden perpetuated error. It is all too easy for some historian to record an erroneous and undocumented statement, and then for subsequent historians, leaning on the earlier work, to repeat the falsehood. Many such erroneous statements have been widely perpetuated over considerable periods of time.

Bonaventura Cavalieri was born in Milan, Italy, in 1598, studied under Galileo, and served as a professor of mathematics at the University of Bologna from 1629 until his death in 1647 at the age of forty-nine. Cavalieri was one of the most influential mathematicians of his time, and the author of a number of works on trigonometry, geometry, optics, astronomy, and astrology. He was among the first to recognize the great value of logarithms and was largely responsible for their

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early introduction into Italy. But his greatest contribution to mathematics was a treatise, *Geometria indivisibilibus*, published in its first form in 1635, devoted to the precalculus *method of indivisibles*—a method that can, like so many things in more modern mathematics, be traced back to the early Greeks, in this case Democritus (ca. 410 B.C.) and Archimedes (ca. 287–212 B.C.). It is quite likely that it was the attempts at integration made by Kepler that directly motivated Cavalieri. At any rate, the publishing of Cavalieri's *Geometria indivisibilibus* in 1635 marks a great moment in mathematics.

Cavalieri's treatise on the method of indivisibles is voluable and not clearly written, and it is not easy to learn from it precisely what Cavalieri meant by an "indivisible." It seems that an indivisible of a given planar piece is a chord of the piece, and a planar piece can be considered as made up of an infinite parallel set of such indivisibles. Similarly, it seems that an indivisible of a given solid is a planar section of that solid, and a solid can be considered as made up of an infinite parallel set of this kind of indivisible. Now, Cavalieri argued, if we slide each member of a parallel set of indivisibles of some planar piece along its own axis, so that the endpoints of the indivisibles still trace a continuous boundary, then the area of the new planar piece so formed is the same as that of the original planar piece, inasmuch as the two pieces are made up of the same indivisibles. A similar sliding of the members of a parallel set of indivisibles of a given solid will yield another solid having the same volume as the original one. (This last result can be strikingly illustrated by taking a vertical stack of cards and then pushing the sides of the stack into curved surfaces; the volume of the disarranged stack is the same as that of the original stack.) These results give the so-called *Cavalieri principles*:

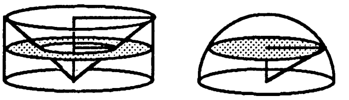
1. *If two planar pieces are included between a pair of parallel lines, and if the lengths of the two segments cut by them on any line parallel to the including lines are always equal, then the areas of the two planar pieces are also equal.*

2. *If two solids are included between a pair of parallel planes, and if the areas of the two sections cut by them on any plane parallel to the including planes are always equal, then the volumes of the two solids are also equal.*

Cavalieri's hazy conception of indivisibles, as sort of atomic parts of a figure, led to much discussion and serious criticism by some students of the subject, particularly by the Swiss goldsmith and mathematician Paul Guldin (1577–1642). Cavalieri recast his treatment in the vain hope of meeting these objections. The French geometer and physicist Gilles Persone de Roberval (1602–1675) ably employed the method and claimed to be an independent inventor of it. The method, or some process very like it, was effectively used by Evangelista Torricelli (1608–1647), Pierre de Fermat (1601?–1665), Blaise Pascal (1623–1662), Grègoire de Saint-Vincent (1584–1667), Isaac Barrow (1630–1677), and others. In the course of the work of these men, results were reached which are equivalent to performing a number of integrations.

The assumption and then consistent use of Cavalieri's second principle can greatly simplify the derivation of many of the volume formulas encountered in a beginning treatment of solid geometry. This procedure has been adopted by a number of textbook writers, and has been advocated on pedagogical grounds. For example, in deriving the familiar formula for the volume of a tetrahedron ($V = Bh/3$), the sticky part is first to show that any two tetrahedra having equivalent bases and equal altitudes on those bases have equal volumes. The inherent difficulty here is reflected in all treatments of solid geometry from Euclid's *Elements* on. With Cavalieri's second principle, however, the difficulty simply melts away.

Accepting Cavalieri's principles as intuitively evident, one can solve many problems in mensuration that normally require the more advanced techniques of the calculus. Let us define two planar pieces that can be placed so that they cut off equal segments on each member of a family of parallel lines, or two solids that can be placed so that they intersect equiareal sections on each member of a family of parallel planes, to be *Cavalieri congruent*. Two figures that are Cavalieri congruent have, of course, equal areas (in the one case) or equal volumes (in the other case). To find the unknown area of some planar piece, or the unknown volume of some solid, one tries to find a comparison figure of easily found area or volume that is Cavalieri congruent to the given figure. It often requires skill to devise an appropriate comparison figure for a given situation. Well known, for example, are appropriate comparison solids for finding the volume of a sphere and then the volume of a spherical ring. In the first case one shows that a hemisphere of radius r is Cavalieri congruent to a circular cylinder of radius and altitude r, from the top



of which a circular cone of radius and altitude r has been removed. In the second case, one shows that a spherical ring obtained by removing from a solid sphere of radius r a cylindrical boring of radius b coaxial with the polar axis of the sphere is Cavalieri congruent to a sphere of diameter equal to the altitude of the spherical ring. In this latter case, it follows that *all spherical rings of the same altitude have the same volume, irrespective of the radii of the rings*. By devising appropriate comparison solids one can find expressions for the volumes of a *hoof* (the figure formed by an oblique plane passing through the center of the base of a right circular cylinder), a *Steinmetz solid* (the solid common to two right circular cylinders of equal radii and having their axes intersecting perpendicularly), a *torus* and many other interesting solids.

The purpose of this article, however, is not to find expressions for certain areas and volumes by use of Cavalieri's principles, but rather to prove two theorems about Cavalieri congruence that at first encounter scarcely seem to be true. It is difficult to believe that a long shallow triangle with a base of a mile length can be Cavalieri congruent to, say, an equilateral triangle of the same area. Nevertheless, we will prove that *any two triangles of the same area are Cavalieri congruent*. Again, since it can easily be shown that there cannot exist a polygon to which a given circle is Cavalieri congruent, it is difficult to believe that there exists a polyhedron to which a given sphere is Cavalieri congruent. A sphere is so round, and a polyhedron is so angular and made up of nothing but planes! Nevertheless, we will prove that *there exists a polyhedron* (actually a tetrahedron) *to which a given sphere is Cavalieri congruent*. We shall give two different proofs of the first theorem.

Theorem 1. *Any two triangles of the same area are Cavalieri congruent.*

First Proof. Let ABC and $A'B'C'$ be the two triangles, with corresponding sides a, b, c, and a', b', c'. We consider three mutually exclusive and exhaustive cases.

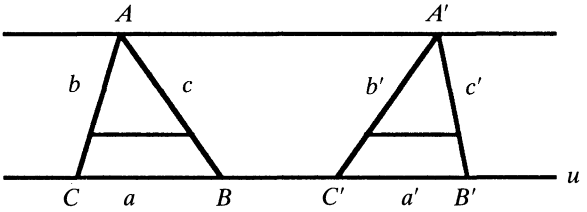


Figure 1

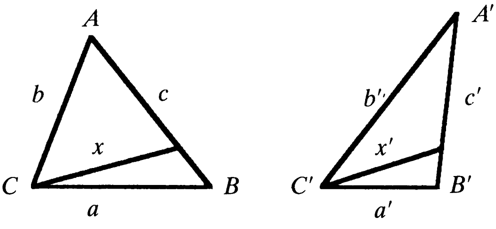


Figure 2

Case 1. (the two triangles have a side of one equal to a side of the other, say $a = a'$).

Place the two triangles so that a and a' lie on a line u, with A and A' on the same side of u (see Figure 1). Since the two triangles have the same area, AA' is parallel to u. Etc.

Case 2. (the sides a and a', b and b', c and c' are related by inequalities, not all of the same sense, say $a > a'$, $b < b'$).

Let cevians x and x' through C and C' divide c and c' similarly (see Figure 2). When $x = a$, then $x/x' = a/a' > 1$. When $x = b$, then $x/x' = b/b' < 1$. Therefore, by the intermediate-value theorem, somewhere between a and b there exists a cevian v such that $v/v' = 1$.

Place the two triangles (see Figure 3) so that v and v' lie on a line u, with vertices A and A' on the same side of u. Since the two given triangles have the same area, so do the two pairs of corresponding subtriangles, and AA' and BB' are parallel to u. Etc.

Case 3. (the sides a and a', b and b', c and c' are related by inequalities, all of the same sense, say $a < a'$, $b < b'$, $c < c'$).

Let m' (see Figure 4) be an interior altitude of triangle $A'B'C'$, say it is through vertex C', and let cevian m through C be such that m and m' divide c and c' similarly. Now $m > m'$, since $m \geq$ altitude on $c > m'$. Let cevians x and x' through C and C' divide c and c' similarly. When $x = m$, then $x/x' = m/m' > 1$. When

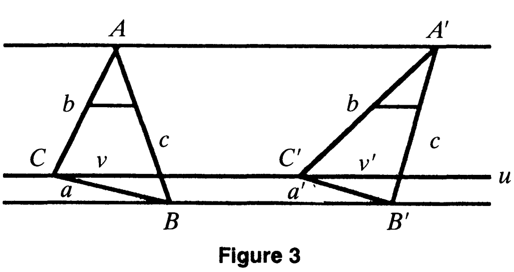


Figure 3

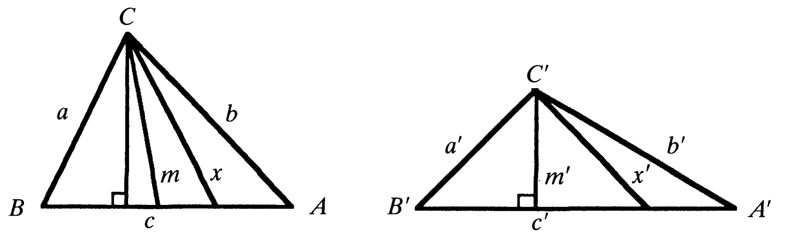


Figure 4

$x = b$, then $x/x' = b/b' < 1$. Therefore, by the intermediate-value theorem, somewhere between m and b there exists a cevian v such that $v/v' = 1$.

Place the two triangles (see Figure 5) so that v and v' lie on a line u, with vertices A and A' on the same side of u. Since the two given triangles have the same area, so do the two pairs of corresponding subtriangles, and AA' and BB' are parallel to u. Etc.

The above proof is a nonconstructive existence proof. That is, though it proves there exists a positioning of the two triangles that establishes their Cavalieri congruence, it does not (except in Case 1) inform us precisely how to effect the desired positioning, for we are not told how to construct the two equal cevians v and v'. We now offer an alternative proof which is constructive.

Theorem 1. *Any two triangles of the same area are Cavalieri congruent.*

Second Proof. If the two triangles ABC and $A'B'C'$ have a side of one equal to a side of the other, we position the two triangles as in Case 1 of our first proof. If the two triangles do not have a side of one equal to a side of the other, first place the two triangles as indicated in Figure 6, where A and A' coincide and BC is parallel to $B'C'$. There is no loss of generality in assuming $BC < B'C'$.

Let BB' and CC' intersect in D, and draw the circle on DA as diameter to cut the line midway between BC and $B'C'$ in M and N. Select either of these two points, say N, and draw DN to cut BC in L and $B'C'$ in L'. Now $LC/L'C' = BC/B'C'$, whence L and L' divide BC and $B'C'$ similarly. Also, since $LN = NL'$ and angle DNA is a right angle, it follows that $AL = A'L'$. Thus AL and $A'L'$ are a pair of equal cevians in the two triangles which divide BC and $B'C'$ similarly. We now position triangles ABC and $A'B'C'$ so that AL and $A'L'$ lie on a line u, with vertices C and C' on the same side of u. Then BB' and CC' will be parallel to u, and the two triangles are in a position that establishes their Cavalieri congruence.

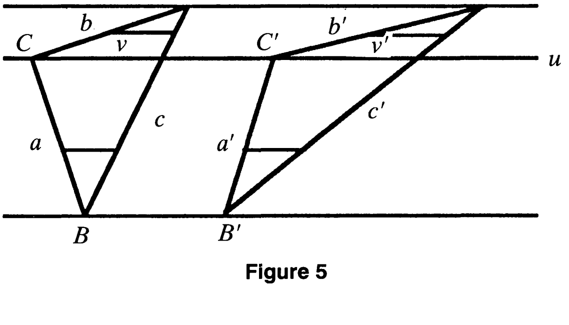


Figure 5

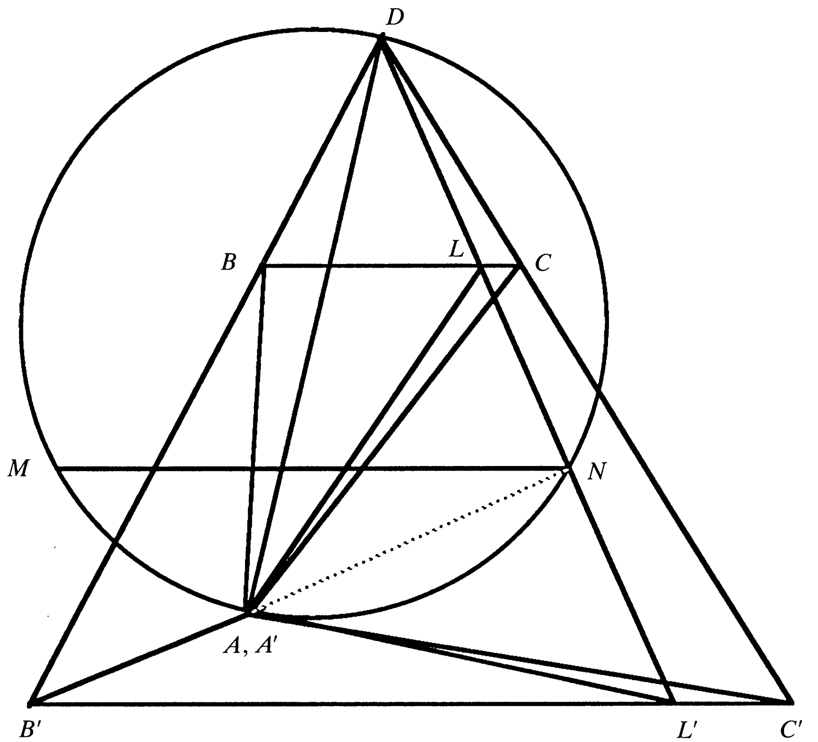


Figure 6

From either of the above proofs of Theorem 1, it is clear that in general two equiareal triangles are Cavalieri congruent in a number of different ways.

We now prove our second theorem; the proof will be a constructive one.

Theorem 2. *There exists a tetrahedron to which a given sphere is Cavalieri congruent.*

Proof. Denote the radius of the given sphere by r. In the planes tangent to the sphere at its north and south poles, draw (see Figure 7) two line segments AB and

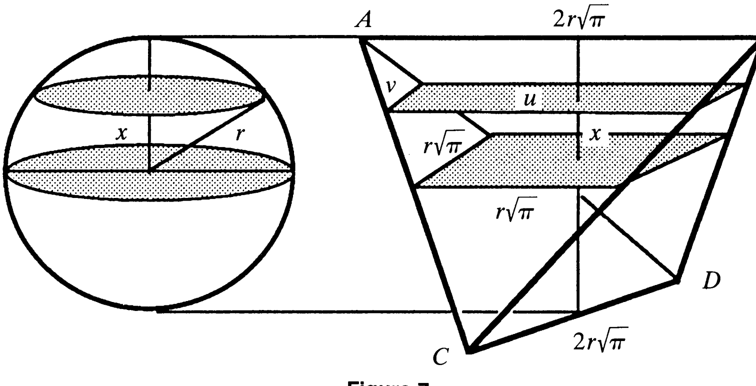


Figure 7

CD perpendicular to one another, each of length $2r\sqrt{\pi}$ and having the line segment joining their midpoints as a common perpendicular.

Form the tetrahedron *ABCD*. The equatorial plane of the sphere cuts the tetrahedron in a square of side $r\sqrt{\pi}$. Let a plane parallel to the equatorial plane and at a distance x from it cut the sphere in a circle and the tetrahedron in a rectangle of sides u and v, where u is parallel to AB and v is parallel to CD. From the figure we see that the circular section of the sphere has area $\pi(r^2 - x^2)$. We also see, from similar triangles, that

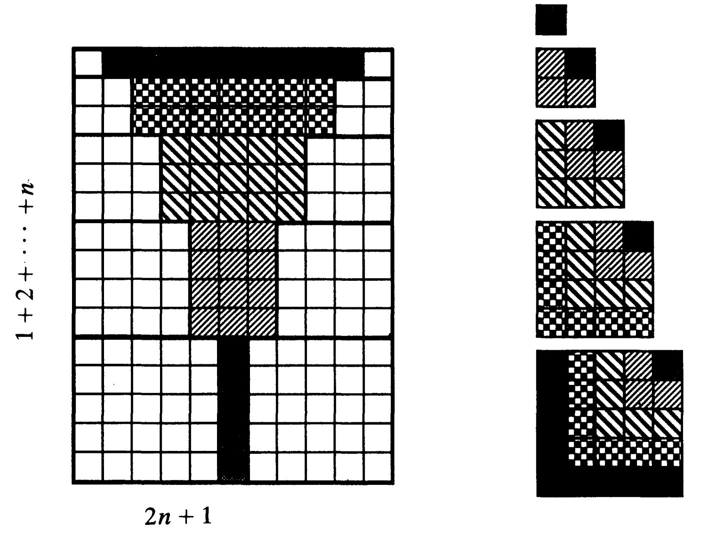
$$\frac{u}{r\sqrt{\pi}} = \frac{r+x}{r}, \qquad \frac{v}{r\sqrt{\pi}} = \frac{r-x}{r},$$

whence

$$uv = \pi(r+x)(r-x) = \pi(r^2 - x^2).$$

It follows that the sphere and the tetrahedron are Cavalieri congruent.

$(1+2+\cdots+n)(2n+1) = 3(1^2+2^2+\cdots+n^2)$



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