Euler and Differentials

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Two recent articles by Dunham [**5**] and Flusser [**10**] have presented examples of Leonhard Euler's work in algebra. Both papers are a joy to read; watching Euler manipulate and calculate with incredible facility is a pleasure. A modern mathematician can see the logical flaws in some of the arguments, yet at the same time be aware that the mind behind it all is that of a unique master.

These two articles reminded me how much fun it is to read Euler. In researching the evolution of the differential a few years ago, I found the work of Euler refreshingly different from that of other seventeenth- and eighteenth-century mathematicians. One can read about Euler's use and misuse of infinite series in most histories of mathematics (e.g. [**2**, pp. 486–490]). This paper offers a glimpse at how Euler used infinitesimals and infinite series to compute differentials for the elementary functions encountered in a typical undergraduate calculus sequence. I hope the reader of this brief survey of Euler's work with differentials will seek out original sources such as [**8**] and [**9**]. As Harold Edwards [**7**] has cogently argued, we have much to learn from reading the masters.

Euler and the 18th Century

Euler (1707–1783) was the most prolific and one of the most influential mathematicians who ever lived. He made major contributions to both pure and applied mathematics and his collected works amount to over 70 volumes. So strong was his influence that historians like Boyer [**2**] and Edwards [**6**] refer to the eighteenth century as the Age of Euler.

Euler made the function concept fundamental in analysis. He saw a function as both any quantity depending on variables and also as any algebraic combination of constants and variables (including infinite sums or products). This is obviously not a modern definition of a function. Still, Euler used his function concept to maximal advantage. As we examine some of Euler's computations, keep in mind the immense insight and unity he achieved with the function approach—a point of view we now take for granted.

In his *Introducio in analysin infinitorum* (1748), one sees the first systematic interpretation of logarithms as exponents. Prior to Euler, logarithms were typically viewed as terms of an arithmetic series in one-to-one correspondence with terms of a geometric series [**3**]. Euler viewed trigonometric functions as numerical ratios rather than as ratios of line segments. He also studied properties of the elementary transcendental functions by the frequent use of their infinite series expansions

[**6**, p. 270]. Euler often used infinite series indiscriminately, without regard to questions of convergence.

Euler's understanding and use of differentials within the framework of functions is the focus of this paper. Before presenting his work, a word about the differential before Euler.

For Leibniz (1646–1716) the differentials dx and dy were, as the name suggests, (infinitesimal) differences in the abscissa x and the ordinate y, respectively [**4**, pp. 70–76]. The infinitesimal was considered to be a number smaller than any positive number. The omission of the "even smaller" higher-order infinitesimals such as $(dx)^2$ or $dx\,dy$, which were deemed negligible relative to dx and dy, was basic to his methods. So powerful were the notation and methods that the differential calculus was truly a *differential* calculus for nearly one and a half centuries: The differential (and not the derivative) was the main object of study.

Leibniz gave other interpretations of the differential, but the mathematicians working in the early eighteenth century tended to favor Leibniz's formulation of a differential as an infinitesimal. It appears in the work of Johann Bernoulli (1667–1748) and in the first calculus textbook, *Analyse des infiniment petits pour l'intelligence des lignes courbes* (1696), which was written by L'Hôpital and which made free use of Bernoulli's ideas (see [**18**, p. 315]). Euler was one of Bernoulli's pupils.

Many of Euler's results and infinite series discussed below were known to Newton, Leibniz, Bernoulli, and others. Euler's work with differentials is unique, however, in his definition of infinitesimals as absolute zeros and in his heavy reliance on infinite series to *develop* his differential calculus.

Differentials as Absolute Zeros

In his *Institutiones calculi differentialis* (1755), Euler stated: "To those who ask what the infinitely small quantity in mathematics is, we answer it is actually equal to zero" [**18**, p. 384]. Euler felt that the view of the infinitesimal as zero adequately removed the mystery and ambiguity of statements such as "The infinitesimal is smaller than any given quantity" or the postulate of Johann Bernoulli that "Adding an infinitesimal to a quantity leaves the quantity unchanged."

Euler then said that the quotient $0/0$ can actually take on any value because

$$n \cdot 0 = 0$$

for all real n and therefore, he concluded,

$$\frac{n}{1} = \frac{0}{0}. \tag{1}$$

He noted that if two zeros can have an arbitrary ratio, then different symbols should be used for the zero in the numerator and the zero in the denominator of the fraction on the right-hand side of equation (1). It is here that Euler introduced the Leibnizian notation of differentials.

Euler denoted an infinitely small quantity by dx. Here $dx = 0$ and $a\,dx = 0$ for any finite quantity a. But for Euler these two zeros are different zeros that cannot be confused when the ratio $a\,dx/dx = a$ is investigated [**18**, p. 385]. In a similar way dy/dx can denote a finite ratio even though dx and dy are zero. "Thus for Euler the calculus was simply the determination of the ratios of evanescent increments—a heuristic procedure for finding the value of the expression $0/0$" [**1**].

The neglect of higher-order infinitesimals was also explained employing quotients. Noting that $dx = 0$ and $(dx)^2 = 0$, where $(dx)^2$ is a zero (or infinitesimal) of second order, Euler reasoned that

$$dx + (dx)^2 = dx$$

because

$$\frac{dx + (dx)^2}{dx} = 1 + dx = 1.$$

By the same reasoning, Euler established that

$$dx + (dx)^{n+1} = dx$$

for all $n > 0$. The omission of higher-order differentials was frequently utilized by Euler in finding the differential dy, where y is a function of x.

Computations with Elementary Functions

The computations discussed in this section are all found in Euler's *Institutiones calculi differentialis*. Their most noteworthy feature is the use of power series expressions for functions from the outset, with no mention of questions of convergence. Thus, whereas in modern textbooks the justification of such infinite series expansions is an advanced topic in differential calculus, for Euler they were the foundation for the calculation of derivatives.

To find dy if $y = x^n$ (n any real number), Euler used the binomial expansion [**9**, p. 99]. If x is increased by an infinitesimal amount dx, then y experiences a change of dy where

$$\begin{aligned} dy &= (x + dx)^n - x^n \\ &= nx^{n-1}\,dx + \frac{n(n-1)}{1}x^{n-2}(dx)^2 + \cdots \\ &= nx^{n-1}\,dx \end{aligned}$$

upon the omission of the higher-order infinitesimals $(dx)^2$, etc. Newton and Leibniz did similar computations for finding the derivative of $y = x^n$, Leibniz using a comparable differential argument while Newton worked with fluxions [**6**, p. 192]. Within the rigorous context of

$$\lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{\Delta x}$$

we all use the essence of this computation (for positive integer powers of x) in our first semester calculus courses.

Euler derived the product rule as follows:

$$\begin{aligned} d(pq) &= (p + dp)(q + dq) - pq \\ &= p\,dq + q\,dp + dp\,dq \\ &= p\,dq + q\,dp \end{aligned}$$

where the last step is due to the omission of the higher-order infinitesimal $dp\,dq$.

Similar computations were done by Leibniz [**4**, p. 143]. This argument is analogous to the proof of the product rule still found in a few present-day textbooks (e.g., [**12**]).

Euler's derivation of the quotient rule is unique in its use of a geometric series [**9**, p. 103]:

$$\begin{aligned}\frac{1}{q+dq} &= \frac{1}{q}\left(\frac{1}{1+dq/q}\right)\\ &= \frac{1}{q}\left(1-\frac{dq}{q}+\frac{dq^2}{q^2}-\cdots\right)\\ &= \frac{1}{q}-\frac{dq}{q^2}.\end{aligned}$$

Then

$$\begin{aligned}d\left(\frac{p}{q}\right) &= \frac{p+dp}{q+dq}-\frac{p}{q}\\ &= (p+dp)\frac{1}{q+dq}-\frac{p}{q}\\ &= (p+dp)\left(\frac{1}{q}-\frac{dq}{q^2}\right)-\frac{p}{q}\\ &= \frac{dp}{q}-\frac{p\,dq}{q^2}\\ &= \frac{q\,dp-p\,dq}{q^2}.\end{aligned}$$

In chapter 6, Euler found the differentials of transcendental functions. For computing the differential of the natural logarithm (which he denoted by the single letter "ℓ" but which we will denote by the usual "log"), Euler used Mercator's series [**9**, p. 122]:

$$\log(1+z) = z-\frac{z^2}{2}+\frac{z^3}{3}-\cdots.$$

Given $y=\log(x)$ then

$$\begin{aligned}dy &= \log(x+dx)-\log(x)\\ &= \log\left(1+\frac{dx}{x}\right)\\ &= \frac{dx}{x}-\frac{(dx)^2}{2x^2}+\frac{(dx)^3}{3x^3}-\cdots\\ &= \frac{dx}{x}.\end{aligned}$$

To illustrate the chain rule, Euler did many examples. For instance, if $y=\log(x^n)$

then letting $p = x^n$ yields $y = \log(p)$, which implies that $dy = dp/p$ where $dp = nx^{n-1}\,dx$. Thus $dy = n\,dx/x$.

Euler's computation of dy for $y = \log(x)$ can be found in a modern nonstandard analysis text [**15**, p. 65]. This may seem unremarkable since nonstandard analysis was developed by Abraham Robinson in the mid-twentieth century to place the notion of infinitesimals and their manipulation on solid logical ground. In fact, it is rare to find nonstandard analysis arguments that are exactly like Euler's, because nonstandard analysis arguments are rarely done in the context of infinite series (see [**11**] and [**16**]).

As an example of Euler's work with trigonometric functions, consider the computation of dy for $y = \sin x$ [**9**, p. 132]. For this purpose he explicitly used the sine and cosine series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \tag{2}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \tag{3}$$

to show that $\sin(dx) = dx$ and $\cos(dx) = 1$. He obtained these results by substituting dx into (2) and (3) and ignoring higher-order differentials. He also employed the trigonometric identity

$$\sin(a + b) = \sin a \cos b + \sin b \cos a. \tag{4}$$

Thus, using (4):

$$\begin{aligned} dy &= \sin(x + dx) - \sin x \\ &= \sin x \cos dx + \sin dx \cos x - \sin x \\ &= \sin x + \cos x\,dx - \sin x \\ &= \cos x\,dx. \end{aligned}$$

This is the most beautifully efficient computation of all those presented, especially when compared to the usual limit computation of the derivative of $y = \sin x$. There one needs to work as follows:

$$\begin{aligned} \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{\sin x \cos(\Delta x) + \sin(\Delta x)\cos x - \sin x}{\Delta x} \\ &= \cos x \lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x} + \sin x \lim_{\Delta x \to 0} \frac{\cos(\Delta x) - 1}{\Delta x} \end{aligned}$$

Then $y' = \cos x$ is obtained using two limits (which must be proven):

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \tag{5}$$

and

$$\lim_{x \to 0} \frac{\cos x - 1}{x} = 0.$$

The first of these limits is captured in Euler's equation $\sin(dx) = dx$. The second limit is comparable to Euler's equation $\cos(dx) = 1$ or $\cos(dx) - 1 = 0$. Although

Euler's derivation is computationally more compact than the standard modern approach, the latter is logically sound. Any method for differentiating the sine function must deal in particular with (5). This is proven geometrically, since in the standard modern approach one defines at the outset the geometric meaning of the trigonometric functions (i.e., cosine and sine parametrize the unit circle). The proof of (5) is relatively easy when compared to the difficulty involved in showing the geometric meaning of the functions Euler defined (without regard to questions of convergence) as the sums of the power series (2) and (3).

In Euler's three-volume *Institutiones calculi integralis* (1768–1770), he defined integration, like Leibniz and Johann Bernoulli, as the formal inverse of the differential. He used the integral symbol and wrote, for example,

$$\int nx^{n-1}\,dx = x^n$$

$$\int dx/x = \log x$$

$$\int \cos x\,dx = \sin x$$

all plus or minus an appropriate constant. The first volume of this work reads like a modern calculus textbook chapter on techniques of integration. Integration by substitution, by parts, by partial fractions, and by trigonometric substitution are all illustrated in a logical and systematic way. Undoubtedly, Euler's well-organized and all-encompassing use of differentials in a function context did much to solidify the popularity of the differential and integral notations on the continent.

The Total Differential

Euler's *Institutiones calculi differentialis* was the first systematic exposition of the calculus of functions of several variables. He understood a function of n variables to be any finite or infinite expression involving these variables. As soon as he introduced these functions, Euler addressed the question of the relationship among the differentials of all the variables involved.

He obtained the result that if

$$V = f(x, y, z)$$

then

$$dV = p\,dx + q\,dy + r\,dz,$$

where p, q, and r are all functions of x, y, and z [**9**, pp. 144–145]. He arrived at this formula in an interesting way. If X is a function of x alone and is increased by an infinitesimal amount dx, then

$$dX = P\,dx$$

by the usual one-variable argument. Similarly, if Y and Z are functions of y alone and z alone respectively, then

$$dY = Q\,dy$$

and

$$dZ = R\,dz.$$

If $V = X + Y + Z$ (i.e., a special function of three variables) then

$$dV = dX + dY + dZ$$
$$= P\,dx + Q\,dy + R\,dz.$$

If $V = XYZ$, then

$$dV = (X + P\,dx)(Y + Q\,dy)(Z + R\,dz) - XYZ.$$

This simplifies (upon omission of higher-order differential terms such as $ZPQ\,dx\,dy$) to

$$dV = YZP\,dx + XZQ\,dy + XYR\,dz.$$

From these two examples, Euler expected that any algebraic expression of x, y, and z has differential

$$dV = p\,dx + q\,dy + r\,dz \tag{6}$$

because a function of three variables can be thought of as a sum of products of these variables. He generalized the result for any number of variables [**9**, p. 146].

Later in the same work, he addressed the concept of partial differentiation [**9**, pp. 156–157]. If y and z are held constant, then by equation (6)

$$dV = p\,dx$$

as there is no change in y or z. (Notice how, for Euler, no change in y is not the same as saying dy is the infinitesimal change in y, even though he defined infinitesimals as being zero.) He then wrote

$$p = (dV/dx),$$

where the parentheses about the quotient remind one that p equals the differential of V (with only the x being variable) divided by dx. Similar meanings apply to $q = (dV/dy)$ and $r = (dV/dz)$. This was Euler's notation and understanding of the concept of partial derivatives. The current symbol ∂ dates from the 1840's [**14**]. Obviously, (6) becomes

$$dV = (dV/dx)\,dx + (dV/dy)\,dy + (dV/dz)\,dz,$$

although Euler did not explicitly write this.

It is worth noting that Euler's exposition of differentials for functions of several variables immediately followed his work with differentials for functions of one variable. Exploring the differential calculus for both single and multivariable functions before passing on to integration is an old idea which I think has merit. It gives the calculus sequence a stronger focus and unity, by concentrating effort on one basic concept (the derivative) in various settings before moving on to its inverse. A recent textbook by Small and Hosack [**17**] takes this approach. Perhaps we will see more of this, especially since computer algebra systems such as *Derive*, *Maple*, and *Mathematica* have taken the pain out of such tasks as surface sketching.

Differentials in Multiple Integrals

Euler frequently let his readers in on his thought processes, even when the procedures seemed fruitless. This was mathematics being done for all to see, not a slick modern textbook treatment. There was no taking down the scaffolding à la Gauss.

Euler, in *De formulis integralibus duplicatis* (1769), gave one of the first clear discussions of double integrals. In the first half of the eighteenth century, $\int\int f(x, y)\,dx\,dy$ denoted the solution of $\partial^2 z/\partial x\,\partial y = f(x, y)$ obtained by antidifferentiation. Euler supplemented this by providing a (thoroughly modern) procedure for evaluating definite double integrals over a bounded domain **R** enclosed by arcs in the xy plane. Euler used iterated integrals:

$$\iint_{\mathbf{R}} f(x, y)\,dx\,dy = \int_a^b dx \int_{f_1(x)}^{f_2(x)} z\,dy,$$

where $z = f(x, y)$. For $z > 0$, Euler saw this as a volume, since $\int z\,dy$ gives the area of a "slice" (parallel to the y-axis) of the three-dimensional region above **R** and under $z = f(x, y)$, and the following integration with respect to x "adds up the slices" to yield the volume [**8**, p. 293]. This is perhaps the first time Leibniz's powerful differential notation was used in tandem with a volume argument employing Cavalieri's method of indivisibles [**2**, p. 361].

Euler also interpreted $dx\,dy$ as an "area element" of **R**. That is, **R** is made up of an infinite set of infinitesimal area elements $dx\,dy$. This is most clearly seen when Euler attempted to change variables [**8**, pp. 302–303]. And it was here that Euler ran into difficulties.

He reasoned that if $dx\,dy$ is an area element and we change variables via the transformation

$$x = x(t, v) = a + mt + v\sqrt{1 - m^2}$$
$$y = y(t, v) = b + t\sqrt{1 - m^2} - mv$$

(a translation by the vector (a, b), a clockwise rotation through the angle α, where $\cos\alpha = m$, and a reflection through the x-axis), then $dx\,dy$ should equal $dt\,dv$. But

$$dx = m\,dt + dv\sqrt{1 - m^2}$$
$$dy = dt\sqrt{1 - m^2} - m\,dv$$

and multiplication gives

$$dx\,dy = m\sqrt{1 - m^2}\,(dt)^2 + (1 - 2m^2)\,dt\,dv - m\sqrt{1 - m^2}\,(dv)^2.$$

Euler rejected this as wrong and meaningless. (How many calculus students wonder, explicitly or implicitly, why we cannot just multiply the differential forms for dx and dy?) Euler decided to attack the problem in a formal non-geometric way, not using area elements but rather by changing variables one at a time (for details, see [**13**]). In this way he arrived at the correct general result:

$$\int\int f(x, y)\,dx\,dy = \int\int f(x(t, v), y(t, v)) \left| \frac{\partial(x, y)}{\partial(t, v)} \right| dt\,dv.$$

In 1899, another great mathematician with a computational flair, Élie Cartan, arrived at the straightforward multiplicative result Euler sought, by using Grassmann's exterior product with differential forms. This is a formal product where the usual distributive laws hold but with the conditions that

$$dx\,dx = dy\,dy = 0$$

and

$$dx\,dy = -dy\,dx$$

(see [**17**, p. 514], and [**13**]). Thus, for Euler's differentials

$$\begin{aligned} dx\,dy &= \left(m\,dt + dv\sqrt{1-m^2}\right)\left(dt\sqrt{1-m^2} - m\,dv\right) \\ &= dt\,dt\,m\sqrt{1-m^2} - m^2\,dt\,dv + (1-m^2)\,dv\,dt - dv\,dv\,m\sqrt{1-m^2} \\ &= -m^2\,dt\,dv + dv\,dt(1-m^2) \\ &= -m^2\,dt\,dv - dt\,dv(1-m^2) \\ &= -dt\,dv. \end{aligned}$$

The minus sign appears because the transformation (involving a reflection) does not preserve orientation. In general, given any transformation from the tv-plane to the xy-plane, the exterior product yields

$$dx\,dy = \frac{\partial(x,y)}{\partial(t,v)}\,dt\,dv.$$

Conclusion

Even in this rudimentary survey of Euler's work with differentials in calculus, it is fascinating to watch a genius grapple with an ambiguous concept (infinitesimal) and attempt to clarify it (absolute zero)—however flawed the attempt. Reading Euler has enriched my teaching of the calculus by keeping me mindful that my students are tackling a subject whose foundations humbled the greatest minds of the past. Even the seemingly fruitless paths can be instructive, as we have seen. It took mathematicians about 150 years to come up with the exterior product for differential forms that Euler needed for the change of variables formula in multiple integrals. How many other Eulerian dead ends may be worth pursuing? Again, the advice of Harold Edwards [**7**] points the way for the teacher and the researcher: "Read the masters!"

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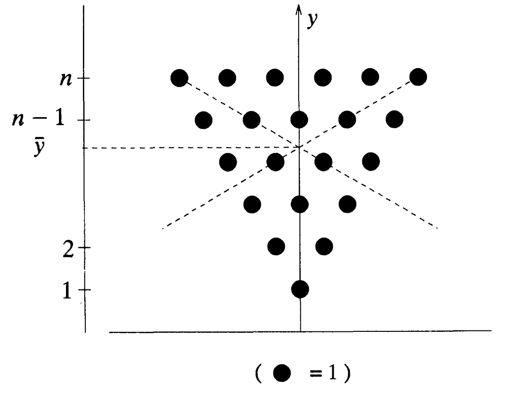
Sum of Squares via the Centroid

$$\bar{y} = 1 + \frac{2}{3}(n-1) = \frac{1\cdot 1 + 2\cdot 2 + \cdots + n\cdot n}{1 + 2 + \cdots + n}$$

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)}{2}\cdot\frac{1}{3}(2n+1) = \frac{1}{6}n(n+1)(2n+1)$$



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