A New Look at an Old Function, $e^{i\theta}$

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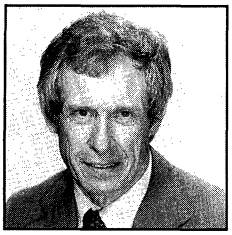

James G. Simmonds studied at the Massachusetts Institute of Technology, earning an S.B. and S.M. in aeronautical engineering (1958) and—after active duty with the U.S.A.F. at NASA, Langley—a Ph.D. in applied mathematics (1965). He joined the Department of Applied Mathematics at the University of Virginia in 1966 and recently served as chair. His research has focused on the theory of shells, but his equally strong interest in pedagogy is reflected in several of his undergraduate textbooks, including *A First Look at Perturbation Theory* (with James Mann; Krieger, 1986) and *A Brief on Tensor Analysis* (2nd ed., Springer, 1993).

Simple geometric concepts and constructions are so important to a beginning student's understanding of complex variables that, as teachers, we should seize every opportunity to exploit this mode of exposition—what we may lose in rigor we surely gain in insight. There is no better way to illustrate this thesis than with Euler's exponential function of purely imaginary argument, $e^{i\theta}$, arguably the most important and useful function in the theory of complex variables.

Most elementary texts define this function by

$$e^{i\theta} \equiv \cos\theta + i\sin\theta \tag{1}$$

and then proceed to establish its properties from those of $\sin\theta$ and $\cos\theta$. This may be satisfactory for students who are comfortable defining functions by power series, but most students visualize the trigonometric functions through their geometric definitions rather than as power series, and from the geometric point of view identifying a combination of trigonometric functions with the exponential $e^{i\theta}$ seems totally unmotivated.

I think the usual approach is backwards. The function $e(\theta)$ whose value is the point on the unit circle with argument θ has simpler properties than $\sin\theta$ and $\cos\theta$, and its geometric definition is more natural. In particular:

- The mysterious addition formulas for $\cos(\theta+\phi)$ and $\sin(\theta+\phi)$ follow immediately from the simple addition formula $e(\theta+\phi)=e(\theta)e(\phi)$, which just expresses the geometric definition of complex number multiplication.
- In contrast to the differentiation formulas for $\sin\theta$ and $\cos\theta$, the validity of

  $$e'(\theta) = ie(\theta) \tag{2}$$

  is obvious geometrically.
- The Taylor expansion (with remainder) for $e(\theta)$—which justifies its identification with $e^{i\theta}$—can, as I show, be given a simple geometric derivation and interpretation. Each term represents a step along an ever-tightening *rectangular spiral* that approaches the point $e(\theta)$, and the bound for the remainder just expresses the fact that the distance along a curve between two points is never shorter than the straight-line distance between these points. Remarkably, *the geometrical construction of the Taylor series does not require knowledge of the derivatives of $e(\theta)$!*

We begin by introducing a Cartesian coordinate system in the Euclidean plane, denoting the origin by O and the point a unit distance away on the x-axis by P. Complex numbers are defined to be vectors (or equivalently points) in the plane, addition is defined by the usual *parallelogram law*, and multiplication of a complex number by a real number is the usual multiplication of a vector by a scalar (a stretch or a shrink, with a reversal of direction if the scalar is negative). All these are standard straightedge and compass constructions.

We define multiplication by the *triangle law* illustrated in Figure 1. The product of two complex numbers Q and R is found by constructing a triangle SOR similar to triangle QOP. Thus, because $|OP| = 1$,

$$|OS| = |OQ| \cdot |OR|, \quad \text{and} \quad \angle SOP = \angle QOP + \angle ROP. \tag{3}$$

That is, to multiply two complex numbers, we multiply their magnitudes and add their angles. Surely this geometric definition is much more revealing to the neophyte than the Cartesian coordinate definition of multiplication, $(u + iv) \cdot (x + iy) = (ux - vy) + i(uy + vx)$. Note that if, as is customary, we identify the complex numbers on the x-axis with the real numbers, the rule for multiplying a complex number by a real number can be considered a degenerate case of the triangle law for the product of two complex numbers.

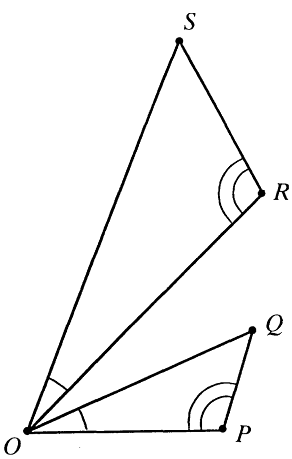

Figure 1
The triangle law of multiplication.

This geometrical definition of multiplication suggests, immediately, that we introduce the polar form of a nonzero complex number. Thus let a ray from the origin through a typical complex number z make an angle θ with the positive x-axis, and let $e(\theta)$ denote the point where this ray intersects the unit circle, as in Figure 2. Then

$$z = |z| e(\theta). \tag{4}$$

The following properties of $e(\theta)$ are then obvious.

- $|e(\theta)| = 1$ because $e(\theta)$ is on the unit circle.
- $e(\theta + 2\pi) = e(\theta)$ because the circumference of the unit circle is 2π.
- $e(\theta) \cdot e(\phi) = e(\theta + \phi)$, the *exponential property*, because to multiply complex numbers of magnitude 1, we just add their angles.

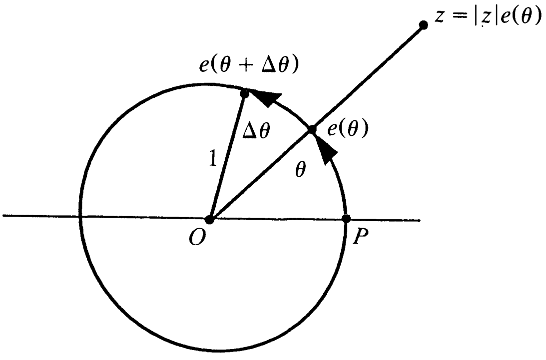

Figure 2
The function $e(\theta)$ as a point on the unit circle.

Now *by definition* $\cos\theta$ and $\sin\theta$ are the real and imaginary parts of $e(\theta)$,

$$\cos\theta + i\sin\theta = e(\theta) \tag{5}$$

and the addition formulas for the cosine and sine are equivalent, in view of the exponential property, to the coordinate rule for multiplication of complex numbers:

$$\begin{aligned} e(\theta)\cdot e(\phi) &= (\cos\theta + i\sin\theta)\cdot(\cos\phi + i\sin\phi) \\ &= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi) \\ e(\theta+\phi) &= \cos(\theta+\phi) + i\sin(\theta+\phi). \end{aligned}$$

From Figure 2 it is also clear that if $\Delta\theta$ is small, then the vector difference $e(\theta+\Delta\theta) - e(\theta)$ is nearly perpendicular to $e(\theta)$—that is, parallel to $ie(\theta)$—and nearly of length $|\Delta\theta|$. Thus the derivative formula

$$e'(\theta) = \lim_{\Delta\theta\to 0}\frac{1}{\Delta\theta}e(\theta+\Delta\theta) - e(\theta) = ie(\theta)$$

is geometrically plausible. We will give a rigorous proof as soon as we have established Taylor's formula for $e(\theta)$.

Thus consider Figure 3, which shows a sequence of points $P_0, P_1, P_2, \ldots$, constructed by means of a sequence of involutes as follows. First, $P_0 = O$, and $P_1 = P$, the point a unit distance from O on the positive x-axis. The point P_2 is found by unwinding a string, initially stretched along the arc of the unit circle from P_1 to $P_\infty = e(\theta)$, until it makes a right angle with the segment P_0P_1. The string, which generates the involute P_2P_∞, is shown in partially unwound position $P_1P_1'P_2'$. Another string, stretched along the involute P_2P_∞, shown in its partially unwound position $P_2P_2'P_3'$, generates an involute P_3P_∞, and so on. Let $s_m(\phi)$ denote the directed distance along the generator $P_m'P_{m+1}'$. Then, as seen from Figure 3, $s_0(\phi) = 1$ and $s_1(\phi) = \phi$. A general formula for $s_m(\phi)$ can be found by the following bit of seventeenth-century style calculus.

Suppose the indicated angle ϕ is increased by an infinitesimal $d\phi$. Then, as shown in the insert in Figure 3, considering the infinitesimal arcs of the involutes to be circular arcs, we find that $ds_{m+1} = s_m(\phi)\,d\phi$. Thus $s_{m+1}(\theta) = \int_0^\theta s_m(\phi)\,d\phi$,

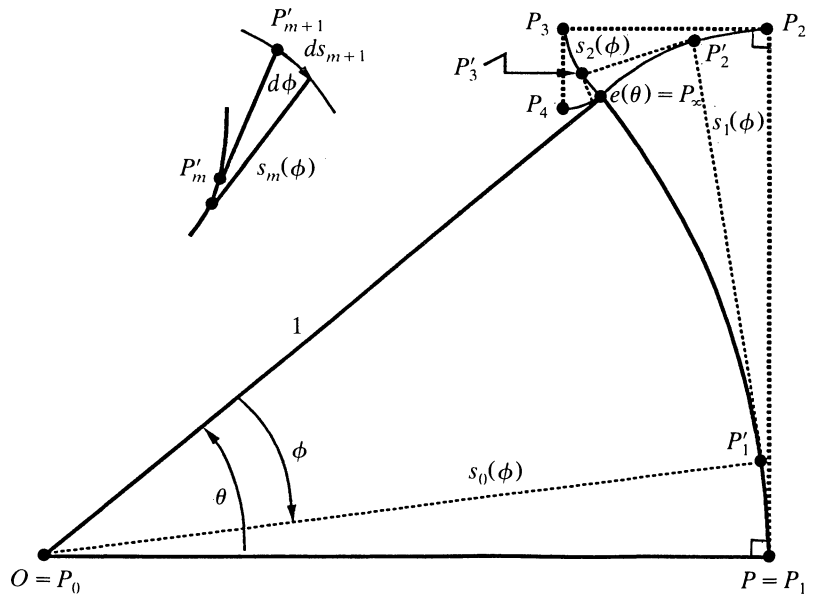

Figure 3
Generators of successive involutes form a rectangular spiral approaching $e(\theta)$.

and since $s_0(\theta) = 1$ it follows by induction that

$$s_m(\theta) = \frac{\theta^m}{m!}, \qquad m = 0, 1, 2 \dots . \tag{6}$$

Recalling that multiplication by i rotates a complex number (vector) counterclockwise through a right angle, we have $P_1 - P_0 = s_0(\theta) = 1$, $P_2 - P_1 = is_1(\theta) = i\theta$, $P_3 - P_2 = i^2 s_2(\theta) = (i\theta)^2/2!, \dots$

$$P_{m+1} - P_m = i^m s_m(\theta) = \frac{(i\theta)^m}{m!}, \qquad m = 0, 1, 2, \dots . \tag{7}$$

If we now start at the origin P_0 and march along the rectangular spiral $P_0 P_1 P_2 \dots P_{n+1}$, the final point P_{n+1} differs from $P_\infty = e(\theta)$ by

$$\begin{aligned} |R_n(\theta)| &= \left| e(\theta) - \left[s_0(\theta) + is_1(\theta) + i^2 s_2(\theta) + \cdots + i^n s_n(\theta) \right] \right| \\ &= \left| e(\theta) - \sum_{k=0}^{n} \frac{(i\theta)^k}{k!} \right| . \end{aligned} \tag{8}$$

The length of the line segment, $|R_n(\theta)|$, is less than the length of the involute $P_{n+1} P_\infty$ joining its endpoints, which is

$$|s_{n+1}(\theta)| = \frac{|\theta|^{n+1}}{(n+1)!}.$$

Thus we have the Taylor formula

$$e(\theta) = 1 + i\theta + \frac{(i\theta)^2}{2!} + \cdots + \frac{(i\theta)^n}{n!} + R_n(\theta), \qquad \text{where } |R_n(\theta)| < \frac{|\theta|^{n+1}}{(n+1)!}. \tag{9}$$

This means that if $P_n(x)$ is the nth Taylor polynomial of the exponential function e^x, then $|e(\theta) - P_n(i\theta)| \to 0$ as $n \to \infty$, which justifies our setting $e(\theta) = e^{i\theta}$.

Note that the Taylor formula with $n = 1$ gives the promised proof of the differentiation formula:

$$\left| \frac{e(\theta + \Delta\theta) - e(\theta)}{\Delta\theta} - ie(\theta) \right| = \left| \frac{e(\theta) \cdot e(\Delta\theta) - e(\theta) - ie(\theta)\Delta\theta}{\Delta\theta} \right|$$

$$= |e(\theta)| \left| \frac{e(\Delta\theta) - 1 - i\Delta\theta}{\Delta\theta} \right| = \left| \frac{R_1(\Delta\theta)}{\Delta\theta} \right| < \frac{|\Delta\theta|}{2},$$

so as $\Delta\theta \to 0$, formula (2) follows.

With the definition of the complex exponential,

$$e^z = e^{x+iy} \equiv e^x e^{iy} \tag{10}$$

—the polar form again!—our discussion is complete. We have introduced the most important transcendental function of a complex variable, using only elementary calculus and emphasizing the simple geometric ideas underlying Euler's function $e^{i\theta}$.

Acknowledgment. My thanks to Paul Warne for preparing the figures.

Nothing New Under the Sun

To the Editor: The "Three-Circle Theorem" by R. S. Hu [*CMJ* 25:3 (May 1994) 211] is just one of a series of statements of concurrence of three lines and collinearity of the points in the configuration formed by three circles, the lines connecting the centers, and the bitangents, given in Section 6–4 ("Three Circles") of my *Plane Geometry and Its Groups* (Holden-Day, 1967). The picture is Figure 6–11; the statement is the first part of (6.31). The statement remains true for lines defined by one internal and two external centers of homothety.

I do not know the first discoverer of the statements; my earliest source is the high school text by August Wiegand, *Dritter Cursus der Planimetrie, enthaltend Lehren der neueren Geometrie für den Schulge brauch* (H. W. Schmidt, Halle, 1871).

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