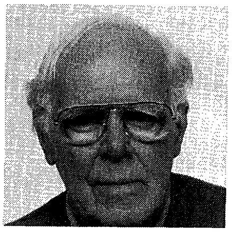
The Asymmetric Propeller

Martin Gardner



For more than twenty-five years **Martin Gardner** wrote the Mathematical Games column in *Scientific American.* The columns are collected in fifteen books, the latest titled *Last Recreations.* He has also written some sixty other books about science, mathematics, philosophy, and literature. His most recent book is *The Night Is Large,* a collection of essays.

The late Leon Bankoff (he died in 1997) was a Beverly Hills, California, dentist who also was a world expert on plane geometry. (For G. L. Alexanderson's interview with Bankoff, see **[1]**.) We became good friends. In 1979 he told me about a series of fascinating discoveries he had made about what he called the asymmetric propeller theorem. He intended to discuss them in an article, but never got around to it. This is a summary of what he told me.

The original propeller theorem goes back at least to the early 1930's and is of unknown origin. It concerns three congruent equilateral triangles with corners meeting at a point as shown shaded in Figure 1. The triangles, which resemble the blades of a propeller, need not form a symmetrical pattern, but may be in any position. They may touch one another or even overlap. Lines *BC*, *DE*, and *FA* are drawn to form a hexagon inscribed in a circle. The midpoints of the three lines mark the vertices of an equilateral triangle.

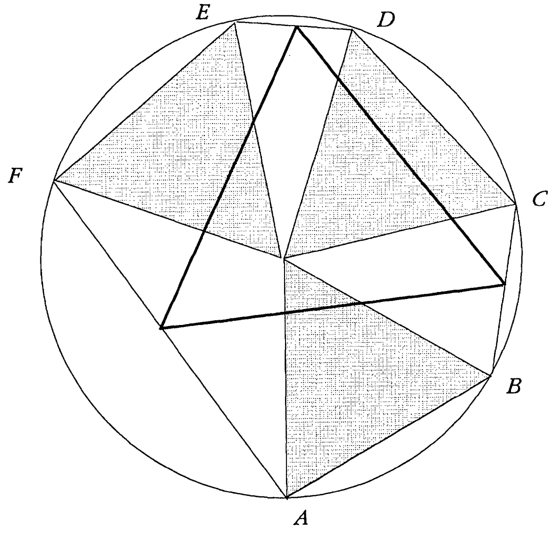


Figure 1

A proof of the theorem, using complex numbers, appeared in **[2]** as the answer to Problem B-1 in the annual William Lowell Putnam Competition. H. S. M. Coxeter sent the proof to Bankoff on a Christmas card, asking him if he could provide a Euclidian proof of the theorem.

Bankoff had no difficulty finding such a proof. In a paper titled "The Asymmetric Propeller" **[3]**, Bankoff, Paul Erdős, and Murray Klamkin made the first generalization of the theorem. They showed that the three equilateral triangles need not be congruent. They can be of any size, as shown in Figure 2, and the theorem still holds. Two proofs are given, one a simple Euclidian proof, the other with complex numbers. As before, and in all subsequent extensions, the triangles may touch one another or even overlap.

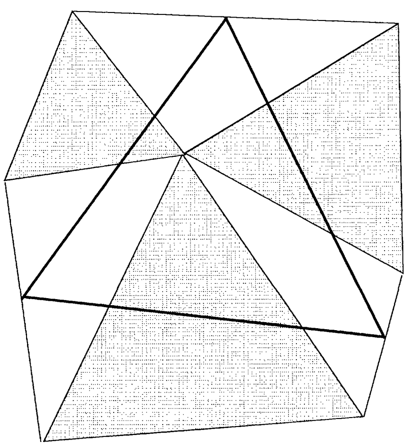


Figure 2

Later, Bankoff made three further generalizations. As far as I know they have not been published.

Second generalization: The propeller triangles need not meet at a point. They may meet at the corners of any equilateral triangle, as shown in Figure 3.

Third generalization: The propeller triangles need not be equilateral! They need only be similar triangles of any sizes that meet at a point. The midpoints of the three added lines will then form a triangle similar to each of the propellers, as shown in Figure 4.

Fourth generalization: The similar triangles need not meet at a point! If the propellers meet at the corners of a fourth triangle of any size, provided it is similar to each propeller, the midpoints of the added lines will form a triangle similar to each propeller. Vertices of the interior triangle must touch corresponding corners of the propellers.

Here is how Bankoff proved his final generalization on a sheet that he typed in 1973. It makes use of Figure 5.

The propellers shown are right triangles, although they can be any type of triangle. Perhaps the proof that follows can be simplified.

If ABC, AHJ, DBE, and FGC are similar triangles, all labeled in the same sense and situated so that corresponding angles meet at the vertices of triangle ABC, then

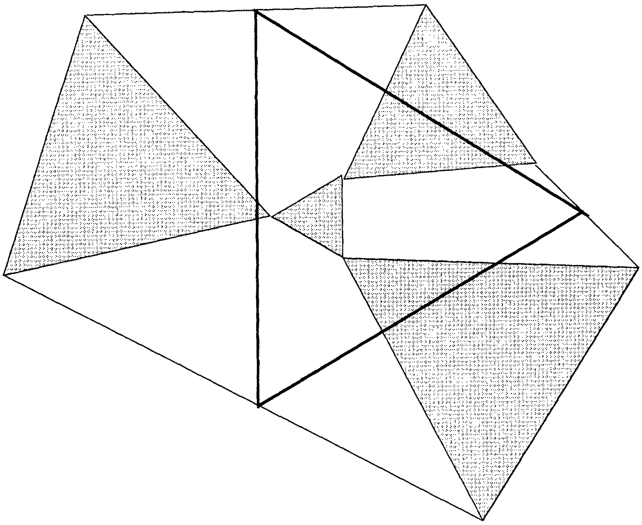


Figure 3

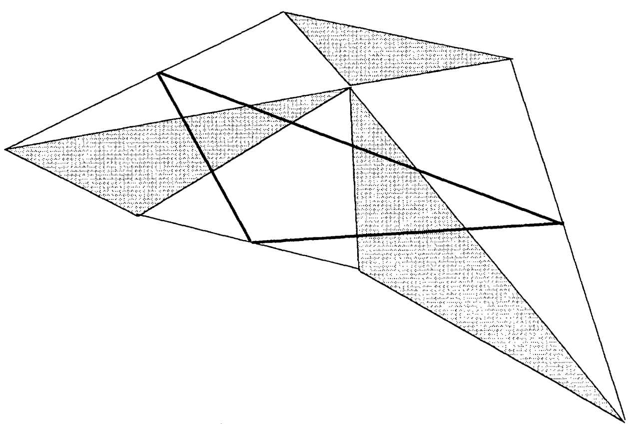


Figure 4

X, Y, Z, the midpoints of DF, GH and JE, are vertices of a triangle similar to the other four.

Proof. Let $\angle BCA = \angle GCF = \alpha$; $\angle DBE = \angle ABC = \beta$; $\angle JAH = \angle CAB = \gamma$; and let P, Q, R, S denote the midpoints of the segments DC, AC, AE and CH respectively. We proceed stepwise to show that triangles PQR, PSZ and finally XYZ are similar to triangle ABC.

If triangle ABD is pivoted about B so that AB falls along BC and DB along EB, it is seen by the relation $AB/BC = DB/BE$ and by the equality of angles ABD and CBE that triangles ABD and CBE are similar and that $EC/AD = BC/AB$, with $\angle EC, AD = \angle BC, BA = \beta$.

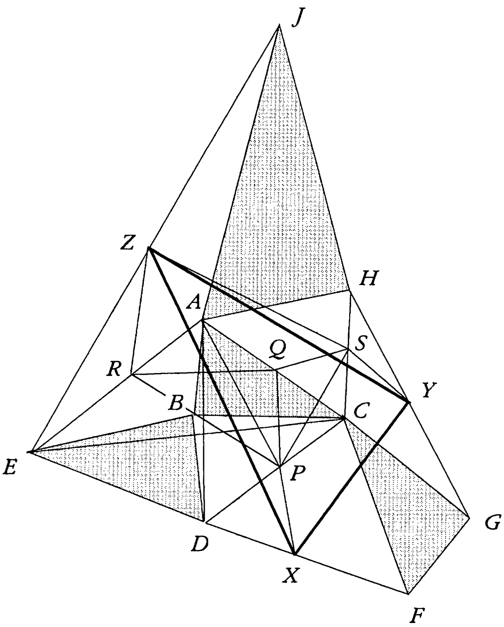


Figure 5

Since RQ is parallel to and equal to half EC while QP is parallel to and equal to half AD, we extend the previous relation to read $RQ/QP = EC/AD = BC/BA$, with $\angle RQ, QP = \angle EC, AD = \angle BC, BA = \beta$. It follows that triangles PQR and ABC are similar.

In like manner, because of the relationship of AJ and AH to RZ and QS as well as to RP and QP in both relative length and in direction, we find triangles ZRP and SQP similar. Then $ZP/SP = ZR/QS = AJ/AH = AC/AB$, with the angles between the segments in the numerator and in the denominator all equal to γ. As a result, triangles PSZ and ABC are similar.

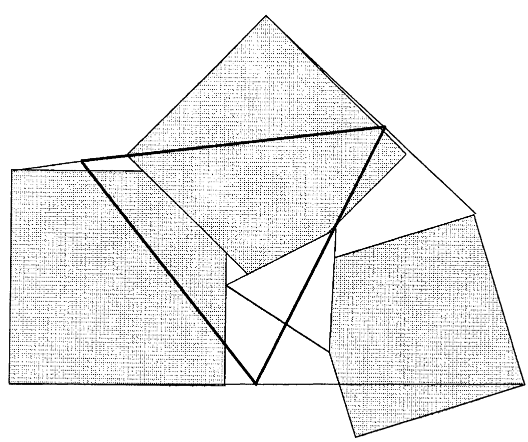


Figure 6

Continuing as before, we find triangles ZPX and ZSY similar since $PX/SY = CF/CB = CA/CG$ and $\angle PX, SY = \angle CF,\ CG = \angle CA, CB = \alpha$.

Noting that in the similar triangles ZPX and ZSY we have $ZX/ZY = ZP/ZS = CA/CB$ and $\angle ZX, ZY = \angle CA, CB = \alpha$, we conclude that triangles XYZ and ABC are similar. ■

And now a question for interested readers to explore. Do the propellers have to be triangles? It occurred to me that if squares are substituted for triangles, as in Figure 6, that equilateral triangle still shows up.

I have written this piece as a tribute to one of the most remarkable mathematicians I have been privileged to know.

References

1. G. L. Alexanderson, A conversation with Leon Bankoff, *College Mathematics Journal* 23 : 2 (1992) 98–117.
2. *American Mathematical Monthly* 75 : 7 (1968) 732–739.
3. Leon Bankoff, Paul Erdős, and Murray Klamkin, The asymmetric propeller, *Mathematics Magazine*, 46 : 5 (1973) 270–272.

Number Theory in the District of Columbia

Although Summers has been in Washington for seven years, in some ways he hasn't fully adjusted to the world outside academia. When his special assistant, Sheryl Sandberg, turned twenty-eight last year, Summers said to her, "Congratulations, you are now a perfect number." She looked at him blankly. "Twenty-eight," he explained. "It's the sum of its divisors. The next perfect number is four hundred and ninety six, and you won't be around then." Sandberg, who met Summers when she was an economics major at Harvard, said, "Larry, who cares about these things?" To which he replied, "How can you not care about these things?"

—John Cassidy, The Triumphalist, *New Yorker* 74 (1998) #18 (July 6, 1998), 57.