Chutes and Ladders for the Impatient

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My 4-year old son likes to play games, but he doesn't have the longest attention span. His younger sister and my wife are less patient. I'm even worse! Recently, while playing *Chutes and Ladders*, I wondered how changing the spinner might shorten the game so that we could complete a game without any of us giving up! To find out, two students and I extended the well-known Markov chain model of *Chutes and Ladders* (e.g., see [**1**] and [**6**]) to investigate the relationship between spinner range and the expected number of turns it takes to complete a game.

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A review of *Chutes and Ladders*

Chutes and Ladders (known as *Snakes and Ladders* in some countries) is a board game popular among the 4-year old set. The game board consists of 100 squares arranged in numerical order, 1 to 100. On each player's turn, the player spins the spinner and moves his or her token the number of squares indicated, if possible. (For example, a player does not move her token if she spins a 2 when on square 99.) The players begin the game off the board, as if there were a square 0. The first player to reach square 100 wins the game.

The chutes and ladders complicate matters. If at the end of a move a player lands at the base of a ladder, then the player climbs to the top of the ladder, thereby getting closer to the goal of square 100. But, if at the end of a move a player lands at the top of a chute, then the player slides down the chute, moving farther from the goal.

The game comes with a spinner which returns the numbers 1 through 6 with equal probability. It would be easy to use a different spinner or to modify the given spinner so that each of the numbers 1 to n is equally likely to occur. We wanted to know which value of n results in the shortest expected number of turns for a player to finish the game. In the next section, we develop a Markov chain model that includes the positions of the chutes and ladders and takes into account the spinner range. At first blush, it may seem that the expected number of turns to complete the game will decrease as n increases, but the requirement that a player land *exactly* on square 100 to end the game (or land on the ladder that goes from square 80 to square 100) means that high values of n result in frequent turns in which a player does not move.

A spinner-range dependent Markov chain for *Chutes and Ladders*

For $i = 1$ to 100, let state i of a Markov chain represent square i in *Chutes and Ladders*. Each player begins the game off the board at state 0. We build a 101×101 transition matrix P for a Markov chain to model n-spinner *Chutes and Ladders* by first assuming that there are no chutes or ladders. Then, the addition of each chute or ladder modifies the transition matrix P by a sequence of simple operations in such a way that the order in which the chutes and ladders are added does not matter. The result is the transformed matrix, P^*, which models the play of the game. This process is demonstrated with a smaller example below.

Recall that entry (i, j) of a transition matrix indicates the probability of transition from state i to state j. For ease of notation, we use the state names, 0 to 100, to represent the row and column numbers, instead of 1 to 101. We assume the spinner range is less than or equal to the number of squares on the board. Hence, for the real game, we consider $n \leq 100$. Because a player can no longer move once he or she has completed the board by landing on square 100, the only entry in row 100 is in column 100. That is, $p_{100,100} = 1$. Once reached, state 100 cannot be left, making it an absorbing state. Further, because state 0 is off the board, column 0 is all 0's, *i.e.*, it is impossible to take a turn and land on state 0. Ignoring the effect of the chutes and ladders, the n-spinner defines the following non-zero transition probabilities or entries in the transition matrix:

- For state $j = 0$ to $(100 - n)$, let $p_{j,j+i} = \frac{1}{n}$ for $i = 1$ to n;
- For $j = (101 - n)$ to 100, let $p_{j,j} = \frac{n-100+j}{n}$; and
- For $j = (101 - n)$ to 100, let $p_{j,j+i} = \frac{1}{n}$ for $i = 1$ to $(100 - j)$.

The resulting matrix contains a diagonal block of entries $\frac{1}{n}$, until a spin can produce an outcome which overshoots square 100, in which case the player does not move on his or her turn. Because it is awkward to display a 101×101 transition matrix, we demonstrate the analysis with the following 10-state variation of *Chutes and Ladders*.

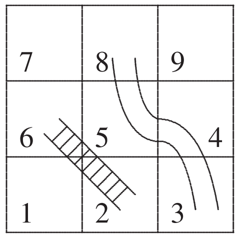

Figure 1. Example game board.

Example. Consider the smaller game board of 9 squares in Figure 1. As in the real game, a player begins in state 0. Assume to begin that the spinner has outcomes 1 through 4 that occur with equal probability. For the time being, ignore any chutes or ladders. A spinner range of 4 results in the 10×10 transition matrix P (see Figure 2) where the game ends when the player reaches square 9. Recall that we denote the rows and columns by numbers from 0 to 9. Rows 5 and 6 of the matrix are identical. From state 5, a spin of 1, 2, 3, or 4 results in the player moving to states 6, 7, 8, or 9, respectively, with equal probability. From state 6, a spin of 1, 2, or 3 results in the player moving to states 7, 8, or 9, respectively, with equal probability of $\frac{1}{4}$. But, a spin of 4 means that the player cannot move (as there is no state 10); hence, the player stays at state 6 with probability $\frac{1}{4}$.

$$P = \begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Figure 2. Example matrix P.

To account for the chutes and ladders, P needs to be modified. Recall that when a player lands at the foot of a ladder, then the player climbs to the top of the ladder. Similarly, when a player lands at the top of a chute, then the player slides to the bottom of the chute. Denote ladders and chutes by $[i, j]$, meaning that if a player lands on square i at the end of the turn, then the player moves to square j (by a ladder if $i < j$ and by a chute if $j < i$). The chutes and ladders from the real game board are: [1, 38], [4, 14], [9, 31], [16, 6], [21, 42], [28, 84], [36, 44], [47, 26], [49, 11], [51, 67], [56, 53], [62, 19], [64, 60], [71, 91], [80, 100], [87, 24], [93, 73], [95, 75], and [98, 78]; there are 19 chutes or ladders.

In general, we construct the transition matrix P^* from P as follows: if $[i, j]$ is a chute *or* ladder, then

1. set $p^*_{i,k} = 0$ for all k (row i is filled with zeros),
2. set $p^*_{k,j} = p_{k,j} + p_{k,i}$ for $k \neq i$ (landing on square i is equated with landing on square j), and
3. set $p^*_{k,i} = 0$ for all k (column i is filled with zeros).

For all other entries (k, l) in P^*, set $p^*_{k,l} = p_{k,l}$. Because row i and column i are replaced by zeros, the matrix P^* can be reduced (by eliminating the rows and columns consisting of all zeros). For the real *Chutes and Ladders* game, this reduction would result in an 82×82 transition matrix $(101 - 19 = 82)$. However, we find it easier not to reduce the matrix in this way, as it makes keeping track of the states more difficult. Let's examine what happens for the smaller board of Figure 1.

Example (continued). The 9-square version of *Chutes and Ladders* has only a single ladder, [2, 6], and a single chute, [8, 3]. Take the ladder first. When a player lands on square 2, she climbs the ladder to square 6. To account for this in the transition matrix P^*, do the following operations in sequence to P:

1. replace row 2 with zeros (because a player is never on square 2),
2. replace column 6 with the sum of column 2 (the probabilities of landing on square 2) and column 6 (the probabilities of landing on square 6), and
3. replace column 2 with zeros (because a player cannot land on square 2).

The chute is handled in a similar fashion. If a player lands on square 8, then he slides down the chute to square 3. To account for this in the transition matrix P^*, do the following operations in sequence to the matrix previously modified by the ladder:

1. replace row 8 with zeros (because a player is never on square 8),
2. replace column 3 with the sum of column 8 (the probabilities of landing on square 8) and column 3 (the probabilities of landing on square 3), and
3. replace column 8 with zeros (because a player cannot land on square 8).

The resulting matrix P^* appears in Figure 3.

$$P^* = \begin{bmatrix} 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{2}{4} & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Figure 3. Example matrix P^*.

Fundamental matrix and game analysis

The Markov chain of *Chutes and Ladders* is absorbing because it has an absorbing state and, by a sequence of moves, it is possible to reach the absorbing state from every other state. An absorbing Markov chain can have its states re-ordered so that it is in block form, called *fundamental form*, with an identity matrix as a lower right block:

$$\left[\begin{array}{c|c} Q & R \\ \hline 0 & I \end{array}\right]$$

Because the absorbing state is the last square on the game board (resulting in the last row and column), P_n^* is already in fundamental form. For the real *Chutes and Ladders*, Q_n is a 100×100 matrix, R_n is a 100×1 matrix, 0 is the 1×100 matrix of all zeros, and I is a 1×1 identity matrix. The block Q_n can be used to determine the expected number of turns to complete the game with an n-spinner. Although we provide a short description of how (and why) this analysis works, more details can be found in [**5, 7, 8**]. Johnson [**7**] used board games to teach Markov chains and provides a description of the fundamental matrix in the context of board games, while Kemeny and Snell [**8**] have a well-written text on Markov chains. The book by Grinstead and Snell also covers Markov chains [**5**, Chapter 11] and is available electronically for free!

The matrix $N_n = (I_{100} - Q_n)^{-1}$ is called the fundamental matrix, where I_{100} is the 100×100 identity matrix. Its entries (i, j) indicate the number of times the player is in state j, given that the player starts in state i. The curious form of N_n is reminiscent of the well-known formula for geometric series: $1 + r + r^2 + \cdots = (1 + r)^{-1}$ for $|r| < 1$. The geometric series of the matrix Q_n converges because the Markov chain is absorbing, so that $N_n = I_{100} + Q_n + Q_n^2 + \cdots = (I_{100} - Q_n)^{-1}$.

Let $\mathbf{1}_{100}$ be the 100×1 vector of all 1's. Then, $N_n\mathbf{1}_{100}$ is a 100×1 vector in which the entry in row i gives the expected number of turns necessary to be absorbed (that is, to transition to state 100) beginning in state i. The entry in row 0 gives the expected number of turns it takes for a player to complete *Chutes and Ladders*, because the player always starts in state 0.

Example (continued). For the 9-square example with a 4-spinner, the matrix in Figure 3 is already in fundamental form. It follows that Q_4 and N_4 are the 9×9 matrices given in Figure 4.

$$Q_4 = \begin{bmatrix} 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{2}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad N_4 = \begin{bmatrix} 1 & \frac{1}{4} & 0 & \frac{89}{60} & \frac{41}{60} & \frac{29}{48} & \frac{193}{144} & \frac{37}{18} & 0 \\ 0 & 1 & 0 & \frac{7}{5} & \frac{3}{5} & \frac{3}{4} & \frac{5}{4} & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{32}{15} & \frac{8}{15} & \frac{2}{3} & \frac{10}{9} & \frac{20}{9} & 0 \\ 0 & 0 & 0 & \frac{4}{3} & \frac{4}{3} & \frac{2}{3} & \frac{10}{9} & \frac{20}{9} & 0 \\ 0 & 0 & 0 & \frac{16}{15} & \frac{4}{15} & \frac{4}{3} & \frac{8}{9} & \frac{16}{9} & 0 \\ 0 & 0 & 0 & \frac{16}{15} & \frac{4}{15} & \frac{1}{3} & \frac{17}{9} & \frac{16}{9} & 0 \\ 0 & 0 & 0 & \frac{16}{15} & \frac{4}{15} & \frac{1}{3} & \frac{5}{9} & \frac{28}{9} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Figure 4.

Multiplying N_4 by the 9×1 vector of all 1's yields

$$\begin{bmatrix} \frac{89}{12} & 7 & 1 & \frac{20}{3} & \frac{20}{3} & \frac{16}{3} & \frac{16}{3} & \frac{16}{3} & 1 \end{bmatrix}^T.$$

Hence, the expected number of turns to complete the 9-square version of the game with a 4-spinner beginning at square 0 is $\frac{89}{12}$. The expected duration is computed for spinner ranges from 1 to 9 in Table 1. Notice that there is more than one local minimum. Increasing the range of the spinner until the expected duration decreases is not sufficient to determine the global minimum. In the example, the expected number of spins to complete the game decreases as the spinner range increases from 2 to 4. Then, a spinner range of 5 increases the expected duration, while a spinner range of 6 results in the (global) minimum expected length of $\frac{43}{6}$ turns to complete the game. The expected length of the game then increases as the spinner range increases from 7 to 9. Notice that the expected duration is the same for spinners of range 8 and 9.

Table 1. The expected number of turns E for one player to complete the 9-square version, as a function of spinner range n.

n	1	2	3	4	5	6	7	8	9
E	∞	$\frac{243}{22} \approx 11.05$	$\frac{173}{23} \approx 7.52$	$\frac{89}{12} \approx 7.42$	7.75	$\frac{43}{6} \approx 7.17$	$\frac{57}{7} \approx 8.14$	9	9

We use the same approach to determine the expected number of turns to complete the real *Chutes and Ladders* (see Table 2). The minimum expected duration occurs with a spinner range of 15. Notice that this data set is close to being single-troughed. The expected number of turns to complete the game decreases as the spinner range is increased from 3 to 15 and then increases as the spinner range increases from 16 to

Table 2. The expected number E of turns for one player to complete *Chutes and Ladders*, as a function of spinner range n.

n	E	n	E	n	E	n	E	n	E
1	∞	21	27.53	41	40.30	61	56.81	81	73.46
2	60.76	22	27.58	42	41.05	62	57.67	82	74.40
3	65.90	23	28.12	43	41.84	63	58.53	83	75.34
4	54.49	24	28.78	44	42.74	64	59.45	84	76.29
5	45.56	25	29.10	45	43.54	65	60.33	85	77.23
6	39.23	26	29.72	46	44.35	66	61.19	86	78.16
7	34.70	27	30.07	47	45.15	67	62.07	87	79.05
8	31.85	28	30.64	48	45.98	68	62.94	88	79.99
9	30.30	29	31.46	49	46.85	69	63.77	89	80.91
10	28.77	30	32.14	50	47.67	70	64.66	90	81.85
11	27.43	31	32.90	51	48.53	71	65.59	91	82.80
12	27.02	32	33.61	52	49.38	72	66.50	92	83.74
13	26.22	33	34.32	53	50.25	73	67.39	93	84.64
14	25.98	34	35.05	54	51.03	74	68.22	94	85.57
15	25.81	35	35.77	55	51.87	75	69.12	95	86.46
16	25.84	36	36.50	56	52.55	76	69.88	96	87.43
17	25.97	37	37.26	57	53.40	77	70.76	97	88.39
18	26.39	38	38.05	58	54.24	78	71.65	98	89.28
19	26.71	39	38.83	59	55.12	79	72.56	99	90.28
20	27.21	40	39.54	60	55.98	80	72.55	100	90.28

79. From 79 to 80, there is a slight decrease in the expected length, but then the game increases for spinner ranges of 81 to 99.

As with the 9-square game, interesting behavior happens for spinner ranges equal to or one less than the board size: the expected duration is the same. Is it always the case for an n-square game that the expected lengths are the same for spinner ranges of $n-1$ and n? Additional calculations suggest that this is the case, but can you prove it or find a counterexample? This is an unsolved problem.

Your turn!

We used the discrete uniform distribution on $\{1, 2, \ldots, n\}$ and considered how changing n affects the expected number of turns to complete *Chutes and Ladders*. There are other natural questions about how to shorten the game: considering non-uniform spinner distributions or different arrangements and lengths of chutes and ladders (see [**1**]; a more abstract question regarding the effect of changing the transitions to a state in a Markov chain is considered in [**3**]). There are endless further questions that can be addressed using Markov chains.

You are invited to modify the Markov chain in this paper and have fun playing—both *Chutes and Ladders*, and with Markov chains. To get started, consider downloading the modifiable Maple file used to compute the expected values in this paper [**2**]. Of course, *Chutes and Ladders* is only one game that can be analyzed by Markov chains. Johnson [**7**] considers *Hi Ho! Cherry-O* as well as *Chutes and Ladders* and provides references to other ways to use Markov chains to analyze games. The Maple file can be edited to help explore the relationship between Markov chains and other board games, too.

Summary. In this paper, we review the rules and game board for *Chutes and Ladders*, define a Markov chain to model the game regardless of the spinner range, and describe how properties of Markov chains are used to determine that an optimal spinner range of 15 minimizes the expected number of turns for a player to complete the game. Because the Markov chain consists of 101 states, we demonstrate the analysis with a 10-state variation with a single chute and single ladder. The resulting 10×10 transition matrix is easier to display and the manipulations are comparable. We conclude with an unsolved problem about expected lengths for generalized *Chutes and Ladders* games.

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