Guards, Galleries, Fortresses, and the Octoplex

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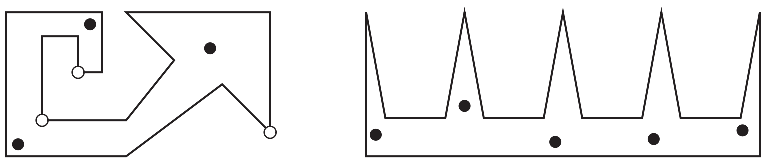
What is the maximum number of guards required to protect a polygonal art gallery with n walls? This unusual geometric problem was posed by Victor Klee in 1973 and is known as the *art gallery problem.* The floor plan of the art gallery is a simple closed polygon, enclosing a simply connected region in the plane. The guards are stationary but can rotate in place to scan the surroundings in all directions. Naturally, they cannot see through walls or around corners. To protect the artwork, every point in the gallery must be visible to at least one guard. A point is visible to a guard provided the segment joining the guard and the point does not pass through the exterior of the gallery. A convex art gallery can be protected by one guard, but galleries with complicated floor plans generally require more guards. It is reasonable to seek the minimum number of guards required to protect a given gallery. The art gallery problem asks for the maximum value of this minimum number among all galleries with n walls.



(a) (b)

Figure 1. Two galleries with 15 walls.

Figure 1 shows two galleries with 15 walls. The gallery in (a) is protected by three guards (black dots). It cannot be protected by fewer guards since no guard can simultaneously see two of the three white vertices. The crown-shaped gallery in (b) requires at least five guards since no two of the five uppermost vertices can be seen by a single guard. It is easy to guard the gallery with five guards, as shown. Some experimentation will convince you that no 15-walled gallery requires more than five guards.

In general, a crown-shaped gallery with t tines has $n = 3t$ walls and requires t guards. The once- and twice-dented crowns with t tines in Figure 2 have $n = 3t + 1$

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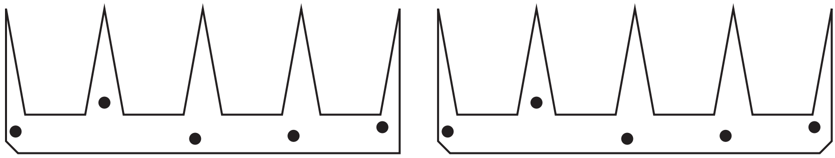

Figure 2. Dented crown galleries.

and $n = 3t + 2$ walls and also require t guards. Thus we have exhibited galleries with n walls that require $\lfloor n/3 \rfloor$ guards for all $n = 3, 4, \ldots$. In 1975, Chvátal [**2**] solved Klee's art gallery problem by proving that no gallery with n walls requires more than $\lfloor n/3 \rfloor$ guards.

Art gallery theorem. *For art galleries with n walls,* $\lfloor n/3 \rfloor$ *guards are sufficient and sometimes necessary.*

Chvátal's proof of sufficiency is intricate, but in 1978 Fisk discovered an ingenious proof that is less sophisticated than Chvátal's and visually appealing. In this article we present Fisk's proof and explain how similar ideas solve two variants of the art gallery problem. We then discuss a guarding problem that cannot be solved by Fisk's method and examine three-dimensional galleries, including a perplexing example, the octoplex, that is not fully protected even if cameras are posted at every corner.

Fisk's colorful idea

Figure 3 illustrates the three steps of Fisk's proof that $\lfloor n/3 \rfloor$ guards suffice to protect any gallery with n walls. First, triangulate the gallery by inserting suitable noncrossing diagonals. A gallery typically has many triangulations; it does not matter which one we use. Second, assign one of three colors (black, gray, and white, say) to each of the n vertices so that every triangle has one vertex of each color. The resulting configuration is a *proper 3-coloring* of the triangulation. In the figure $n = 12$, and there are three black, four gray, and five white vertices. Finally, post guards at the three

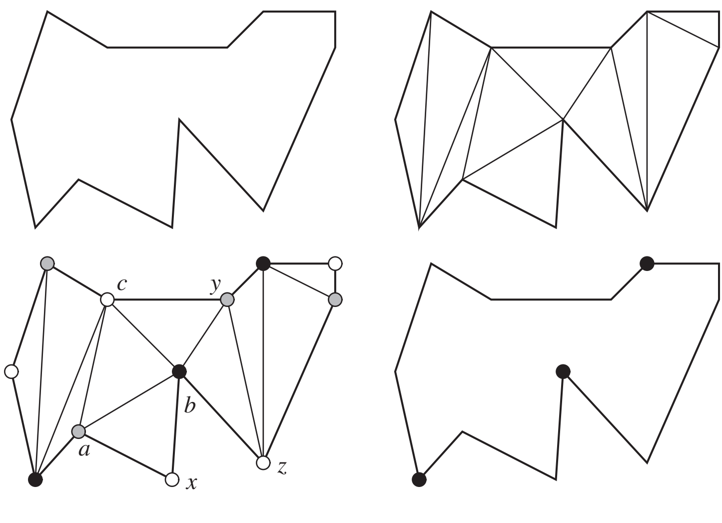

Figure 3. Fisk's proof of the art gallery theorem.

black vertices. Every triangle is certainly protected—since every triangle has a black vertex—and hence the entire gallery is protected. The four gray vertices or the five white vertices also protect the gallery, but the black vertices give us the fewest guards in this case.

The same argument applies to any gallery with n walls. In a proper 3-coloring of a triangulation the average number of vertices per color is $n/3$. Thus some color occurs at most $n/3$ times. Because the number of vertices of each color must be an integer, the color used least often occurs at most $\lfloor n/3 \rfloor$ times. Guards at these vertices protect the entire gallery.

Triangulations

Fisk's argument relies on the following result.

Theorem. *A triangulation of a polygon possesses a proper* 3*-coloring.*

A direct construction explains why. Begin by assigning three different colors to the vertices of any triangle in the triangulation. The colors for the remaining vertices are then determined by the requirement that every triangle contain vertices of each color. For instance, start by assigning three different colors to triangle abc in Figure 3, as shown. Vertex x must be the same color as c. Also, y must be the same color as a, and then z must be the same color as c. Continue in this manner until the entire triangulation is properly 3-colored.

We have *also* assumed that every polygon can be triangulated. This can be proved by induction using the following fundamental result, whose proof is not as easy as one would expect [**6, 9**].

Theorem. *A polygon with more than three sides has a diagonal.*

More art gallery problems

The art gallery theorem has inspired work on related problems in which the rules are altered in two ways to make the model more realistic or more interesting: either restrict or relax the allowed shapes of the galleries; or alter the powers or responsibilities of the guards. Such variants are called *art gallery problems.* We always seek the maximum number of guards required among all galleries with n walls. Art gallery problems are also called *illumination problems;* each guard is a light source, and the goal is to illuminate the entire gallery. The results and algorithms here have applications in robotics and distributed sensor networks (e.g., [**3**], [**13**]).

Right-angled galleries

In a *right-angled gallery* adjacent walls meet at right angles, like the floor plans of most buildings. A right-angled gallery can be drawn so that the walls alternate between horizontal and vertical. It follows that the number of walls must be even.

The comb-shaped galleries in Figure 4 play the role of the crown-shaped galleries. Each tooth of the comb adds four more walls and requires an additional guard. These galleries show that some right-angled galleries with n walls require $\lfloor n/4 \rfloor$ guards. (The dented comb on the right in Figure 6 treats the cases when $n - 2$ is divisible by 4.)

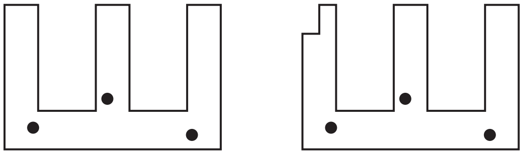

Figure 4. Comb-shaped galleries.

To show that $\lfloor n/4 \rfloor$ guards suffice, we modify Fisk's argument. Figure 5 illustrates a promising strategy. First, partition the right-angled gallery with n walls into quadrilaterals by inserting non-crossing diagonals. The resulting configuration is a *quadrangulation* of the gallery. Then assign four colors to the vertices so that each quadrilateral has one vertex of each color. In this *proper* 4-*coloring* the least frequently used color occurs at most $\lfloor n/4 \rfloor$ times. Post guards at these vertices (the black ones in Figure 5). It seems we have established the following result.

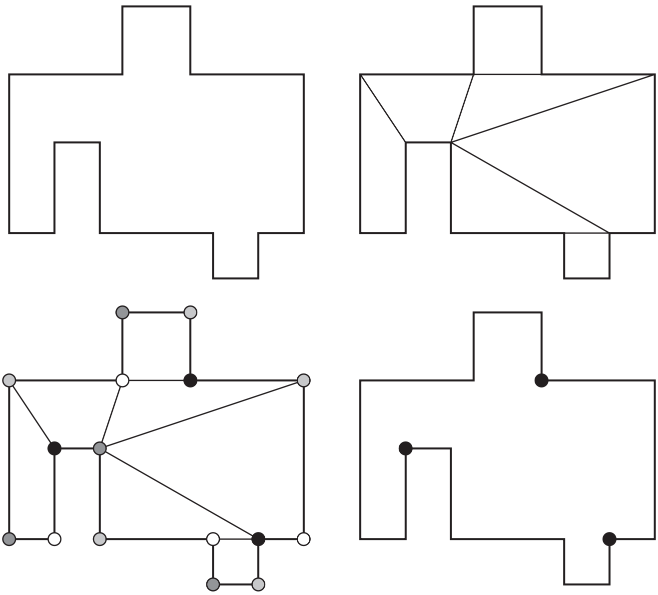

Figure 5. Guarding right-angled galleries.

Right-angled art gallery theorem. *For right-angled art galleries with n walls,* $\lfloor n/4 \rfloor$ *guards are sufficient and sometimes necessary.*

Unfortunately, our coloring argument sometimes fails to post the guards correctly, as in Figure 6. The guards at the three black vertices fail to protect part of the upper nook of the gallery. To guarantee that the argument works, we need a *convex* quadrangulation—one whose quadrilaterals are all convex. In 1985, Kahn, Klawe, and Kleitman [**5**] eliminated the flaw in the argument by proving the following difficult result.

Convex quadrangulation theorem. *Every right-angled art gallery has a convex quadrangulation.*

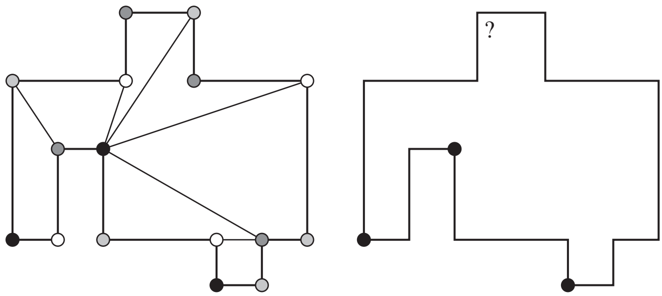

Figure 6. A convex quadrangulation is required.

A mighty fortress

In the *fortress problem* we prevent a surprise attack on a polygonal fortress by posting guards on its perimeter. Every point *outside* the fortress must be visible to a guard along a direct sight-line that avoids the interior of the fortress. The goal is to find the maximum number of guards required among all fortresses with n walls. It is not difficult to show that a convex fortress with n walls requires $\lceil n/2 \rceil$ guards.

O'Rourke and Wood (see Chapter 6 of [**10**]) found a way to adapt Fisk's coloring argument to prove that $\lceil n/2 \rceil$ guards suffice to protect a fortress with n walls. Their method turns the fortress inside out, exchanging the roles of the interior and exterior.

Figure 7 illustrates the steps for a particular fortress with 11 walls. We begin by forming the convex hull—the smallest convex polygon containing the fortress. The *hull pockets* (shaded gray) are the polygons outside the fortress but inside the convex hull. Triangulate each hull pocket. Place a new vertex v far away from the fortress and join it to each vertex on the convex hull of the fortress using non-crossing segments and curves, as shown. Split a particular hull vertex a to produce two nearby vertices a_1 and a_2, apportioning the edges at a between a_1 and a_2 so that v and the hull pockets belong

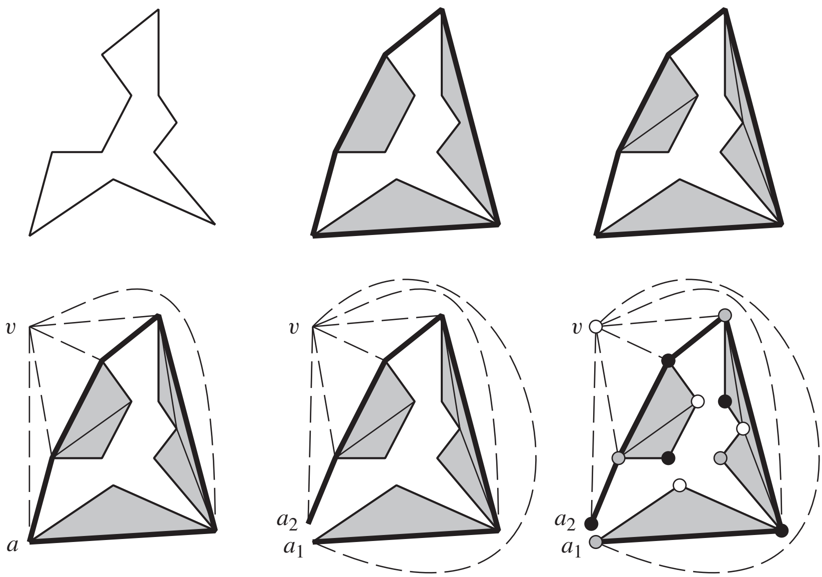

Figure 7. Turning the fortress inside out.

to a single bounded region, as shown; both a_1 and a_2 are joined to v. The interior of the fortress is now part of the unbounded region. Find a proper 3-coloring of the $n+2$ vertices of the bounded region. A proper 3-coloring exists because the bounded region can be unfolded to form a triangulation of a polygon with $n+2$ vertices (see Figure 8). Without loss of generality vertex v is white. In Figure 7 there are five black, four gray, and four white vertices.

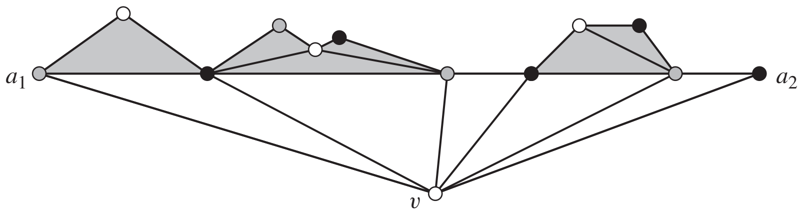

Figure 8. The unfolded fortress is a triangulated polygon.

We claim that guards at the gray vertices cover the entire exterior of the fortress once we merge a_1 and a_2 to restore the original fortress. Because each vertex of the convex hull is joined to the white vertex v, the vertices of the convex hull alternate between gray and black in a chain from a_1 to a_2. This implies that the gray vertices cover the exterior of the convex hull of the fortress. The gray vertices also cover the hull pockets since each hull pocket is triangulated, and each triangle has a gray vertex. Similarly, guards at the black vertices protect the fortress. (See Figure 9.) Because the alternating gray-black chain of vertices from a_1 to a_2 has at most $n+1$ vertices, one of these two colors occurs at most $\lfloor (n+1)/2 \rfloor = \lceil n/2 \rceil$ times. This solves the fortress problem.

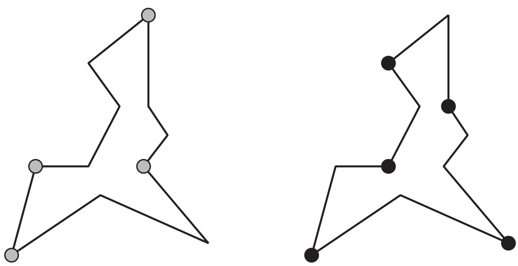

Figure 9. Guards at the gray or the black vertices protect the fortress.

Fortress theorem. *For fortresses with n walls,* $\lceil n/2 \rceil$ *guards are sufficient and sometimes necessary.*

Half-guards

Here is an art gallery problem that cannot be solved by coloring. Suppose we want to protect a gallery with stationary cameras, each with a fixed 180° field of vision. We call this type of camera a *half-guard*. When we post a half-guard, we must also specify the direction it faces, as in Figure 10. The half-guard problem asks for the maximum number of half-guards needed to protect a gallery with n walls. It seems likely that

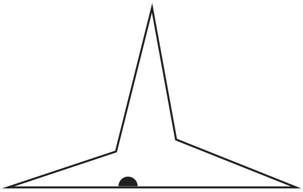

Figure 10. A gallery protected by a half-guard along a wall.

more than $\lfloor n/3 \rfloor$ half-guards will be needed to compensate for the restricted field of vision. Surprisingly, this is not the case.

Half-guard theorem. *Any art gallery with n walls can be protected by* $\lfloor n/3 \rfloor$ *half-guards.*

The use of non-vertex half-guards is unavoidable. For instance, no vertex placement of a lone half-guard does the job for the gallery in Figure 10. Because we must sometimes post half-guards away from vertices, a vertex-coloring argument cannot work.

Tóth [**15**] proved the half-guard theorem in 2000 using a complicated extension of Chvátal's inductive proof of the original art gallery theorem. The proof leads to an algorithm that posts at most $\lfloor n/3 \rfloor$ half-guards at vertices, along walls, and sometimes even in the interior of any gallery with n walls. Whether interior guards are actually required remains in doubt, which prompted Tóth to pose this conjecture.

Conjecture. *Any art gallery with n walls can be protected by* $\lfloor n/3 \rfloor$ *half-guards posted only at vertices or along walls.*

The octoplex

The *three-dimensional* art gallery problem asks for the maximum number of cameras needed to protect the interior of a polyhedron with a fixed number of vertices or walls. We use cameras instead of guards since it may be necessary to post them on the ceiling in mid-air or at other inconvenient locations. We make the simplifying assumption that a camera can see in all directions. This is a natural art gallery problem that remains unsolved.

Guards posted at each vertex of a two-dimensional gallery certainly protect the whole gallery. Astonishingly, this assertion is false for three-dimensional galleries. The counterexample in Figure 11 is constructed as follows [**7**]. Start with a 20-by-20-by-20 cube. Remove a rectangular channel 12 units wide and 6 units deep from the center of the front face. There is an identical channel in the back face. The channels in the left and right faces are 6 units wide and 3 units deep, while those in the top and bottom faces are 6 units wide and 6 units deep. The polyhedron that remains is the *octoplex*. It consists of eight 4-by-7-by-7 theaters connected to one another and to a central lobby by passage-ways 1 unit wide. The octoplex has 56 vertices and 30 walls.

The center point b of the octoplex is not visible to a camera at vertex a because the line segment from a to b exits and then re-enters the octoplex, as indicated. Similar reasoning shows that b is hidden from cameras at all other 55 vertices, too. This means that even if we post a camera at every vertex, part of the octoplex is unprotected.

Art gallery problems in three dimensions are fundamentally different from those in two dimensions. No formula is known for the maximum number of cameras needed

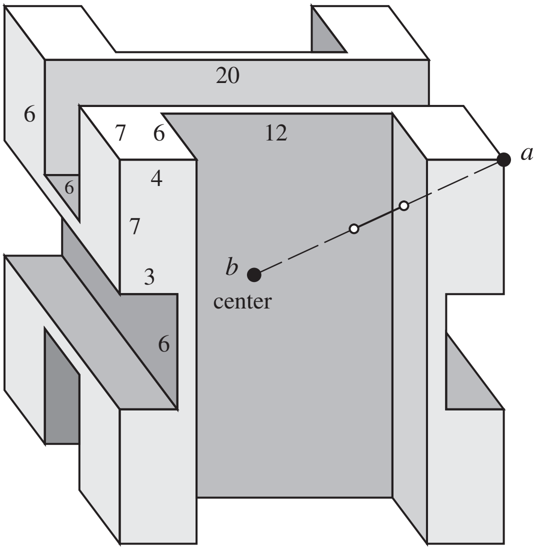

Figure 11. Cameras at every vertex do not protect the octoplex.

to protect a three-dimensional gallery with a given number of vertices or walls. Finding such a formula is believed to be very difficult. Even getting good *bounds* for the number of guards needed for three-dimensional galleries would be a significant breakthrough. See Chapter 10 of [**10**] for details.

Three-dimensional fortresses: research problems for students

In the three-dimensional fortress problem the goal is to post guards on the surface of a polyhedron (the fortress) so that every point in the exterior of the polyhedron is visible to at least one guard. We restrict our attention here to *convex* polyhedra. Even these present interesting problems.

Some observations simplify the types of guard configurations we need to consider: a configuration of guards that protects the faces of a convex polyhedral fortress also protects the entire exterior; if a convex polyhedral fortress can be protected by g guards, then it can be protected by g guards at suitable vertices. The first observation follows from convexity. The second observation is true because a non-vertex guard enlarges its field of vision by moving to a suitable vertex.

A convex polyhedron can be represented as a *plane graph* in which each region (including the unbounded region) corresponds to a face of the polyhedron. To protect the fortress we need to post guards at vertices of the plane graph so that every face has at least one guard. Of course, we want to use as few guards as possible. Figure 12 shows that two guards protect a tetrahedron; four guards protect an octagonal prism; and two guards protect a pentagonal bipyramid. It is easy to verify that these fortresses cannot be protected by fewer guards.

In 1983, Grünbaum and O'Rourke [**10**] found a bound for the number of guards in terms of the number f of *faces* of the fortress, namely, $\lfloor (2f-4)/3 \rfloor$. They constructed a polyhedron with f faces for which $\lfloor (2f-4)/3 \rfloor$ guards are necessary for each $f \geq 5$; however, their proof of sufficiency requires $f \geq 10$.

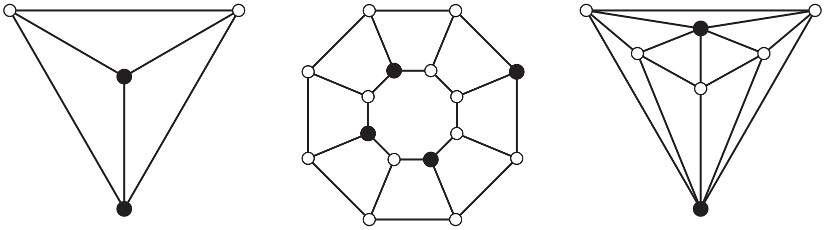

Figure 12. A tetrahedron, an octagonal prism, and a pentagonal bipyramid.

Problem 1. *Do* $\lfloor(2f-4)/3\rfloor$ *guards suffice for* $5 \leq f \leq 9$?

Problem 2. *Is there a simpler proof of sufficiency that is valid for all* $f \geq 5$?

In 1996, Szabó and Talata [**14**] (also see [**1**]) found a bound for the number of guards in terms of the number v of *vertices* of the fortress, namely,

$$\left\lfloor \frac{v}{4} \right\rfloor + \left\lfloor \frac{v+1}{4} \right\rfloor.$$

Let g be the minimum number of guards needed to protect a given convex polyhedral fortress with $f \geq 10$ faces and v vertices. We know that

$$g \leq \left\lfloor \frac{2f-4}{3} \right\rfloor \quad \text{and} \quad g \leq \left\lfloor \frac{v}{4} \right\rfloor + \left\lfloor \frac{v+1}{4} \right\rfloor.$$

Neither of these upper bounds is always better than the other (see Table 1).

Table 1. Upper bounds on the number of guards

polyhedral fortress	f	v	$\lfloor \frac{2f-4}{3} \rfloor$	$\lfloor \frac{v}{4} \rfloor + \lfloor \frac{v+1}{4} \rfloor$	g
octagonal prism	10	16	5	8	4
pentagonal bipyramid	10	7	5	3	2

Problem 3. *Find new bounds on the number of guards required to protect a convex polyhedral fortress. Such bounds could involve both* f *and* v*, say, or still other parameters.*

Further reading

For more information about art gallery theorems see the recent popular book by Michael [**7**], the classic book by O'Rourke [**10**], the update by Shermer [**12**], and the survey by Urrutia in the handbook on computational geometry [**11**]. Coloring arguments figure prominently in [**1**], [**8**], and the survey by Żyliński [**16**].

Summary. The art gallery problem asks for the maximum number of stationary guards required to protect the interior of a polygonal art gallery with n walls. This article explores solutions to this problem and several of its variants. In addition, some unsolved problems involving the guarding of geometric objects are presented.

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