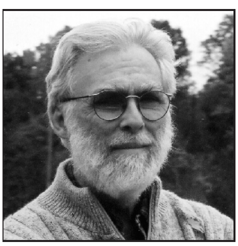
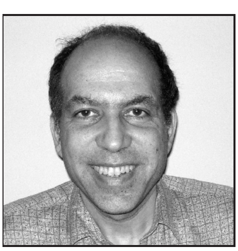
Straw in a Box

Richard Jerrard, Joel Schneider, Ralph Smallberg, and John Wetzel

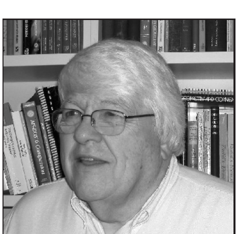
Richard Jerrard (jerrard@uiuc.edu) received his Ph.D. from the University of Michigan, spent most of the subsequent 48 years at the University of Illinois (Urbana, IL 61801), and is now an emeritus professor. He started in applied mathematics and gradually migrated to topology and geometry. His recent work is with John Wetzel on "fitting" problems, of which this article is an example.



Joel Schneider was Vice President for Education and Research at Sesame Workshop (formerly Children's Television Workshop). He was the content director for Square One TV, the Workshop's long running televised series on mathematics, and was the architect of many of its other nationally and internationally broadcast programs in mathematics and science. In 1992, he received the Communications Award from the Joint Policy Board for Mathematics. Regrettably, Joel died on September 12, 2004, as this article was being prepared.[1]



Ralph Smallberg (rsmallberg@mindspring.com) is a science education advisor to Sesame Workshop and a developer of science and mathematics curriculum, software, and television programming for children. Among the widely used programs to which he has contributed as author, designer, or content director are *Square One TV*, *3-2-1 Contact*, and *The Voyage of the Mimi*.



John Wetzel (j-wetzel@uiuc.edu) earned his Ph.D. at Stanford in 1964 after undergraduate work at Purdue. His entire academic career was spent at the University of Illinois, from which he retired in 1999. He shares a condo in Champaign with his wife Rebecca and a miscellany of hippopotamic *objet d'art*, including an aggregation of some sixty-three hippo cookie jars. Always interested in classical geometry, he has most recently been studying the ways in which one shape can fit into another—problems he regards as fitting for retirement.

[1] To our knowledge, amid the furor provoked by the 2003 Regents exam, Joel Schneider first spotted the unintended problem with a straw in a box that we address here, reflecting as ever the care he took when presenting mathematics to children—and his nose for a good problem.

Introduction

A crisis occurred recently in New York after an unexpectedly large number of high school seniors failed a state-wide exit examination in mathematics. This incident serves as a cautionary example of the turmoil that may result from high-stakes testing in the secondary grades. The test at issue, the June 2003 Mathematics A Regents Examination, was subsequently widely condemned by school officials and other educators for what they perceived to be its inappropriate difficulty and unsound construction. For an overview of this episode, see Goodenough [**3**], Sullivan [**7**], and Hoff [**4**]. A copy of the examination and its scoring key can be found at [**6**]. Let us be clear from the start that it is not our intention here to critique this examination. Our interest lies in a different direction.

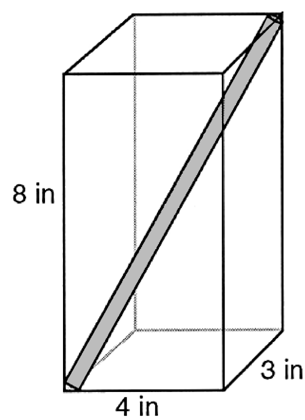


Figure 1. The problem of interest.

Problem 34, one of several controversial problems on the Regents exam, is reproduced in Figure 1. The answer included in the scoring key,

$$\sqrt{3^2 + 4^2 + 8^2} = \sqrt{89} \approx 9.4, \text{ correct to the nearest tenth,}$$

makes it clear that the test makers intended to ask for the length of the space diagonal of a rectangular box of the given (interior) dimensions. The determination of this length requires two applications of the Pythagorean theorem or one application of the formula for the distance between two points in space in terms of their coordinates. Critics complained that the problem makes unreasonable demands on the reading skills of the students, it requires too highly developed spacial visualization skills, it lies beyond the scope of the Math A curriculum, and it tests *aptitude* rather than knowledge. Similar criticisms were directed at other problems in the test.

34 A straw is placed into a rectangular box that is 3 inches by 4 inches by 8 inches, as shown in the accompanying diagram. If the straw fits exactly into the box diagonally from the bottom left front corner to the top right back corner, how long is the straw, to the *nearest tenth of an inch*?

But read Problem 34 again! What, precisely, is the question? The problem asks for the length of a straw that can be placed "exactly" inside the box, and it explicitly directs the reader's attention to a straw drawn with a small but appreciable diameter lying along the space diagonal of the box. Test-savvy students probably glossed over the inherent ambiguity in the problem wording and illustration, dismissed the straw entirely as a figurative device, and correctly intuited that the Regents examiners sought the length of the longest *line segment* that fits within the box having the given dimensions.

Less test-savvy students, however, may just as plausibly have concluded that a straw in the box was the central premise of the problem. Their intuition may accordingly have led them to an altogether different and significantly more difficult problem: determine the longest straw as pictured that can fit diagonally within the given box.

On behalf of students who may have been disposed to read too closely and look too carefully for their own good, we consider this more difficult problem.

A few preliminaries

Introducing suitable notation, we pose the problem as follows:

> Let a_1, a_2, and a_3, be given positive numbers and r a given small nonnegative number. Find the length l of the longest right circular cylinder of radius r that fits diagonally in the box with edges a_1, a_2, and a_3.

Take Cartesian coordinates so that one corner O of the box is the origin (0, 0, 0) and the diagonally opposite corner P has coordinates (a_1, a_2, a_3). Let $M(m_1, m_2, m_3)$ be the midpoint of the diagonal segment OP, so that $m_i = \frac{1}{2}a_i$ for each i.

If a right circular cylinder S_0 with radius r and length l fits in this box, then the right circular cylinder S_1 formed by reflecting S_0 through the midpoint M also lies in the box; and the right circular cylinder S that lies midway between S_0 and S_1 has the same radius r and length l as S_0 and S_1, it is centered at the point M, and it follows from convexity that it lies in the box. We call such a cylinder *central*. Consequently *when searching for the longest cylinder of given radius that fits diagonally in the box we need to consider only central cylinders.* This simplifies our subsequent work considerably.

Let's shorten "central right circular cylinder" to "straw." Standard compactness arguments show that for each sufficiently small positive r there is a longest straw of radius r that fits in the box along the diagonal OP. We call such a straw *maximal*.

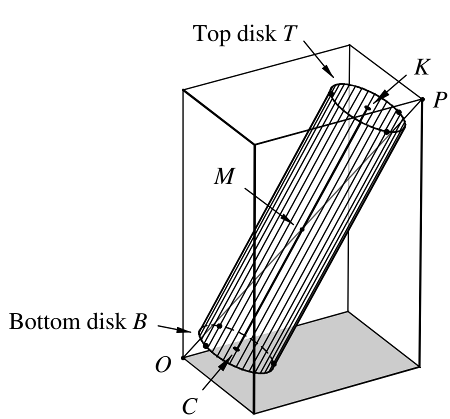


Figure 2. Maximal central straw in a box.

Let S be a maximal straw that fits in the box along the diagonal OP, and let B be the bottom (base) disk of S near O, which we call the *bottom* of S, and T the top disk of S near P, which we call the *top* of S (Figure 2). Evidently B and T must each touch at least one face of the box, or the straw could be lengthened. Since S is central, the configurations of the bottom disk B near O and of the top disk T near P are exactly the same, that is to say, the situation near P is completely symmetric with respect to the point M to the situation near O. If the bottom B touches only one or only two coordinate planes near O, then the top T touches the one parallel face plane or the two parallel face planes near P; and in each of these cases a small rotation about a suitable axis through M would move S entirely into the interior of the box, where it could be lengthened—contrary to our assumption that it is as long as possible. We have established the following fundamental result. (See Figure 2.)

Lemma. *The bottom B of the maximal straw S that fits diagonally in the box is tangent to each of the three coordinate planes of the box that meet at O, and similarly the top T of S is tangent to the three face planes of the box meeting at P.*

Penny in a corner redux

Finding the locus of the center of a disk of given radius r in the first octant that is tangent to all three coordinate planes was posed as the "penny in a corner" problem on the 1948 Putnam Competition (see Mackey [**5**]). A solution can be found in Gleason, Greenwood, and Kelly [**2**, pp. 256–57]; but for the sake of completeness and to formulate matters in the form we need, we present a somewhat different solution (cf. Alexander and Wetzel [**1**]). As we shall soon see, the locus turns out to be the portion Σ_r of the sphere of radius $r\sqrt{2}$ that lies within the cube with one vertex at the origin and opposite vertex at the point (r, r, r) (see Figure 3); more precisely, the locus is the surface-with-boundary,

$$\Sigma_r = \left\{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 = 2r^2 \quad \text{and} \quad 0 \le x_i \le r, \quad i = 1, 2, 3\right\}.$$

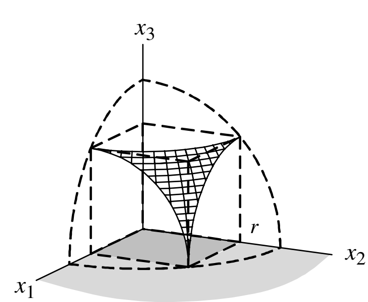


Figure 3. The surface Σ_r.

The key geometric fact is the following. Suppose a plane π with unit normal $\mathbf{n} = (n_1, n_2, n_3)$ passes through a point $C(c_1, c_2, c_3)$ in the first octant and meets the x_1x_2-plane in a line ℓ (Figure 4). Let X be the foot of the perpendicular from C to the plane $x_3 = 0$ and Y the foot of the perpendicular from C to the line ℓ (Figure 4). Writing α for the angle between the normal $\mathbf{n}$ and the vertical direction, we see that $\angle XYC = \alpha$, and it follows that

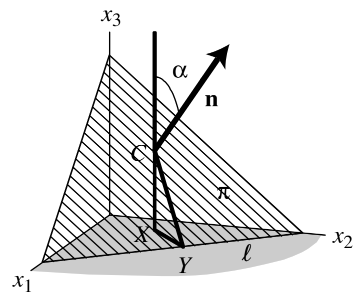


Figure 4. The key fact.

$$
\begin{aligned}
c_3 &= CX = CY \sin\alpha \\
&= CY \cdot \sqrt{1-\cos^2\alpha} \\
&= CY\sqrt{1-n_3^2}.
\end{aligned}
\tag{1}
$$

Now suppose a disk D of radius r and center C lies in the first octant and is tangent to all three coordinate planes, as pictured in Figure 5. If the plane π of the disk D meets each coordinate plane in a line, then C is at distance r from each such line, and formula (1) holds in each direction. It follows that

$$C = (c_1, c_2, c_3) = \left(r\sqrt{1-n_1^2}, r\sqrt{1-n_2^2}, r\sqrt{1-n_3^2}\right), \tag{2}$$

and solving for $\mathbf{n}$ gives the normal to the plane π of the disk D:

$$\mathbf{n} = (n_1, n_2, n_3) = \left(\pm\frac{1}{r}\sqrt{r^2-c_1^2}, \pm\frac{1}{r}\sqrt{r^2-c_2^2}, \pm\frac{1}{r}\sqrt{r^2-c_3^2}\right). \tag{3}$$

Note that these formulas are trivially true in the limiting cases in which the disk actually lies in a coordinate plane and the normal $\mathbf{n}$ points in a coordinate direction.

The desired result is an immediate consequence (see Figure 3).

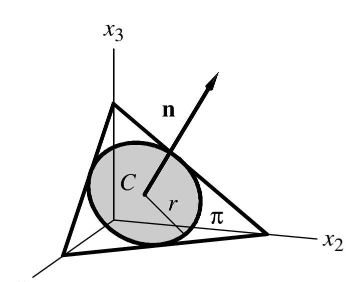


Figure 5. Penny in a corner.

Theorem (Penny in a corner). *Let r be a given positive real number. The center of a disk of radius r in the first octant that is tangent to all three coordinate planes lies on the surface Σ_r.*

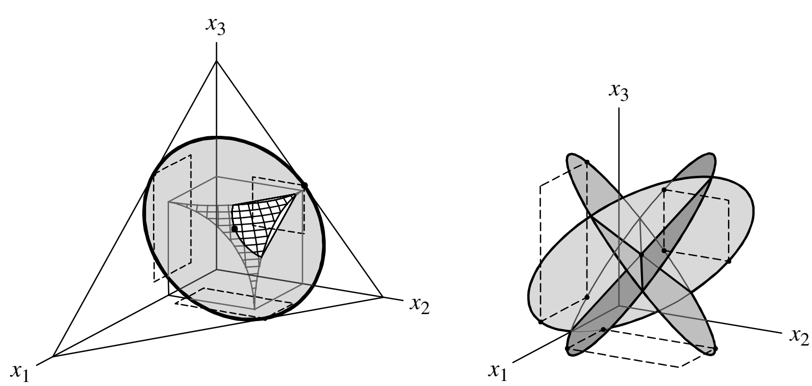
Proof. From (2) we see that $0 \le c_i \le r$ because $|n_i| \le 1$, and

$$c_1^2 + c_2^2 + c_3^2 = 2r^2 \tag{4}$$

because **n** is a unit normal. ■

The geometry of the situation is a bit obscure, so we elaborate a little. First, although the center of the disk (Figure 5) lies on the surface Σ_r, the disk itself is generally *not* tangent to this surface (for example, see Figure 6(a)).

The boundary of Σ_r comprises three (open) quarter-circular arcs σ_1, σ_2, and σ_3 together with their three endpoints, $(0, r, r)$, $(r, 0, r)$, and $(r, r, 0)$ (see Figure 3). If C is one of these three endpoints, then there is exactly one disk of radius r with center C that is tangent to the coordinate planes, and if C lies on one of the boundary arcs σ_i then there are two such disks. But if C is an interior point of Σ_r, then there are exactly *four* different disks with center C and radius r that are tangent to all three coordinate planes, pictured in Figure 6. The normals to the planes of these disks are given by choosing the appropriate signs in (3) (see [**2**, pp. 256–57]).



(a) The important disk. (b) The other three disks.

Figure 6. The four disks.

It is an interesting exercise to find the coordinates of the points of tangency of these disks with the coordinate planes and to show that the points of contact in each coordinate plane form a rectangle whose sides are parallel to the coordinate axes, as pictured.

In particular, each point C of Σ_r is the center of exactly one disk of radius r that is tangent to all three coordinate planes and whose normal **n** has nonnegative components. This disk is the one that plays a role as the bottom of the straw in our problem.

Diagonal straw in a box

Returning to our problem, we suppose that S is the maximum straw of given radius r that lies diagonally in the given box. The center $C(c_1, c_2, c_3)$ of its bottom B lies on the surface Σ_r. The center K of the top disk T of S has coordinates $(a_1 - c_1, a_2 - c_2, a_3 - c_3)$ because S is centered, and it follows that the length l of the straw is

$$
\begin{aligned}
l = CK &= \sqrt{(a_1 - 2c_1)^2 + (a_2 - 2c_2)^2 + (a_3 - 2c_3)^2} \\
&= 2\sqrt{(m_1 - c_1)^2 + (m_2 - c_2)^2 + (m_3 - c_3)^2}.
\end{aligned}
$$

But the vector $\overrightarrow{CK}$, the axis of the straw, is normal to the plane of the bottom disk B. If we write

$$t = \tfrac{1}{2}l = CM = MK,$$

we see that $\overrightarrow{CK} = l\mathbf{n} = 2t\mathbf{n}$, or in coordinates

$$m_i - c_i = \frac{t}{r}\sqrt{r^2 - c_i^2},$$

for each i. This gives a quadratic equation for c_i,

$$c_i^2(r^2 + t^2) - 2r^2 m_i c_i + r^2(m_i^2 - t^2) = 0, \tag{5}$$

and the quadratic formula gives

$$c_i = \frac{r}{r^2 + t^2}\left(m_i r + \varepsilon_i t\sqrt{r^2 - m_i^2 + t^2}\right), \tag{6}$$

where $\varepsilon_i = +1$ or -1.

For each choice of signs $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, substituting (6) into equation (4) (and removing a factor r^2) gives an equation for t in terms of r, m_1, m_2, and m_3,

$$\sum_{i=1}^{3}\left(\frac{1}{r^2 + t^2}\left(m_i r + \varepsilon_i t\sqrt{r^2 - m_i^2 + t^2}\right)\right)^2 = 2,$$

that, in principle, can be solved for t in terms of r and the dimensions of the box. This equation can be written

$$F(t) = 2rt\sum_{i=1}^{3}\varepsilon_i m_i\sqrt{r^2 + t^2 - m_i^2} - \left(2r^4 - t^4 + r^2t^2 + \tfrac{1}{4}(t^2 - r^2)d^2\right) = 0, \tag{7}$$

where $d = OP = 2\sqrt{m_1^2 + m_2^2 + m_3^2}$. Squaring three times shows that t^2 is a root of a polynomial equation of degree at most 16 whose coefficients are polynomials in the given geometric data. Once the root t is known, one can substitute into (6) to find the coordinates of the center C. The coordinates of the center K of the top are then easily computed, and the problem is solved.

Unfortunately, equation (7) is too complicated to solve explicitly for t in terms of r and (m_1, m_2, m_3). This makes determining the maximum length l as an *explicit* function of the given geometric data out of the question.

At this level of generality, a number of questions remain. For $\varepsilon_i = +1$ the root of the quadratic (6) is certainly positive, but for $\varepsilon_i = -1$ it is positive if and only if $t \le m_i$. Considerable numerical experimentation suggests that $c_i < 0$ when $\varepsilon_i = -1$ in (7), but we have not been able to establish this rigorously.

For $t \ge 0$, the function $F(t)$ is defined for

$$t \ge \sqrt{\lfloor \max\{m_1^2, m_2^2, m_3^2\} - r^2 \rfloor},$$

and its graph is increasing both for t near this lower endpoint and also for large t, when the function grows like t^4. Since the distance between the bottom center C and the corner O is $r\sqrt{2}$, we also have the inequality

$$\tfrac{1}{2}d - r\sqrt{2} \le t < \tfrac{1}{2}d. \tag{8}$$

It follows that the root of (7) that we seek lies in the range

$$\max\left\{\sqrt{\lfloor \max(m_1^2, m_2^2, m_3^2) - r^2 \rfloor}, \tfrac{1}{2}d - r\sqrt{2}\right\} \le t < \tfrac{1}{2}d; \tag{9}$$

this supplies a starting interval for algorithms for solving (7) numerically. The behavior of the function on this interval is difficult to discern because the function is complicated, but the geometry of the problem suggests that there should be only one root t in this range.

The Regents problem

Returning to the Regents question, recall that the box in Problem 34 (Figure 1) is $3'' \times 4'' \times 8''$, and we estimate that the straw, as drawn, has diameter $0.4''$. With these data, equations (6) become

$$\begin{aligned} c_1 &= \frac{1}{0.04 + t^2}\left(0.06 + \varepsilon_1 \frac{t}{5}\sqrt{t^2 - 2.21}\right) \\ c_2 &= \frac{1}{0.04 + t^2}\left(0.08 + \varepsilon_2 \frac{t}{5}\sqrt{t^2 - 3.96}\right) \\ c_3 &= \frac{1}{0.04 + t^2}\left(0.16 + \varepsilon_3 \frac{t}{5}\sqrt{t^2 - 15.96}\right), \end{aligned} \tag{10}$$

and the key equation (7) is

$$\begin{aligned} F(t) = 400t^4 - 8916t^2 + 354.72 + 240\varepsilon_1 t\sqrt{t^2 - 2.21} \\ + 320\varepsilon_2 t\sqrt{t^2 - 3.96} + 640\varepsilon_3 t\sqrt{t^2 - 15.96}. \end{aligned} \tag{11}$$

Numerical investigations for the various sign choices $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, show that only the choice $(+, +, +)$ leads to a bottom center C that lies in the positive octant.

Making this choice of signs, we seek the root of (11) that according to (9) lies between

$$\tfrac{1}{2}\sqrt{89} - (0.2)\sqrt{2} \approx 4.434 \quad \text{and} \quad \tfrac{1}{2}\sqrt{89} \approx 4.717.$$

The graph of this function over this interval is shown in Figure 7. The zero is

$$t \approx 4.498,$$

correctly rounded to three places. The centers of the bottom and top disks turn out to be $C(0.191, 0.183, 0.100)$ and $K(2.809, 3.817, 7.900)$, and the length is approximately $8.995''$, which to the nearest tenth of an inch is $9.0''$. This is the answer to the most literal reading of the problem if we take $r = 0.2''$.

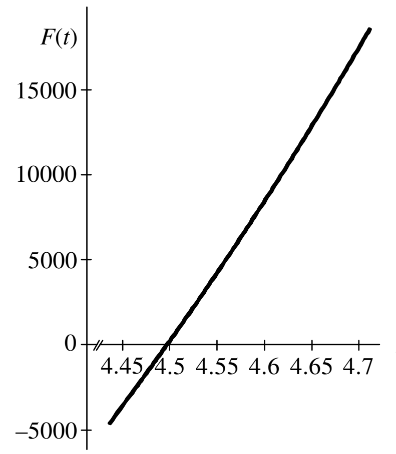


Figure 7. The graph of $F(t)$.

One might wonder how large the radius of the straw can be if its length, properly rounded, is to give the Regents answer of 9.4″. This would require that the length of the longest straw lie in the range between 9.350 and $\sqrt{89}$, and for this to be true, as one can see from equation (11), the radius has to be less than 0.037″.

The general problem

More generally, of course, one could ask for the length l of the longest right circular cylinder of given radius r that fits in the $a_1 \times a_2 \times a_3$ box in any way whatsoever, not necessarily along the space diagonal. The solution of this problem (which to our knowledge is unsolved), would be a function

$$l = f(r, a_1, a_2, a_3)$$

defined on an interval $0 \leq r \leq r_{\max}$ giving the length l of the desired right circular cylinder. Since the diameter of a box is the length of its space diagonal, evidently

$$f(0, a_1, a_2, a_3) = \sqrt{a_1^2 + a_2^2 + a_3^2},$$

provided we regard the space diagonal as a (degenerate) right circular cylinder of radius $r = 0$. If similarly we regard a disk as a right circular cylinder of length 0, it seems clear that

$$f(r_{\max}, a_1, a_2, a_3) = 0.$$

The radius $r_{\max}$ of the largest disk that fits in an $a_1 \times a_2 \times a_3$ box has recently been determined (see Alexander and Wetzel [**1**]), so the interval $[0, r_{\max}]$ on which f is defined is known in terms of the given dimensions of the box, and the value of f is known at each end of the interval. Beyond the obvious fact that f is a decreasing function of r, little is known about it.

Still more generally, one could ask for a *necessary and sufficient condition* along the lines of Post [**8**], who in answer to a 1964 question of Steinhaus gave a set of 18 inequalities on the six sides of two triangles whose disjunction is necessary and sufficient for the first triangle to fit within the second (that is, if any one of the 18 inequalities holds, then first triangle fits in the second, and, conversely, if the first triangle fits in the second, then at least one of the 18 inequalities holds). So one could seek a necessary and sufficient condition on the positive reals a_1, a_2, a_3 and the nonnegative reals r and l for a right circular cylinder of radius r (a line segment in the extreme case $r = 0$ and a disk in the extreme case $l = 0$) to fit in a box of edge lengths a_1, a_2, and a_3. To the best of our knowledge, this problem also remains unsolved.

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Watch Your Units!

Stan Wagon (wagon@macalester.edu) of Macalester College (St. Paul, Minn.) spotted an ad for a Dodge Sprinter Westfalia van in the June/July 2005 issue of *National Geographic Adventure*, in which this was the only text:

2 WEEKS VACATION
8 HOLIDAYS
52 WEEKENDS
5 "SICK DAYS"
———————
119 DAYS OFF