CLASSROOM CAPSULES

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Classroom Capsules consists primarily of short notes (1–3 pages) that convey new mathematical insights and effective teaching strategies for college mathematics instruction. Please submit manuscripts prepared according to the guidelines on the inside front cover to the new Editors Ricardo Alfaro and Steven Althoen, University of Michigan-Flint, Flint, MI 48502.

Distortion of average class size: The Lake Wobegon effect

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Universities love to boast about their presumably small average class size. And how often are parents stunned to hear how large their freshman's average class size seems to be? And since they are using the same words, we can safely assume that they are reporting the same thing. Can't we?

Actually, no! There is a distorting phenomenon that occurs when computing average class sizes that needs to be widely circulated. Namely, the average reported by the university, which is the same as the average experienced by the faculty, is significantly different from the average experienced by the students. Rather than try to explain this at the abstract level, let me present a hypothetical example.

A certain (remarkably small) university has precisely 200 students. All 200 are taking the same five courses, say English, Mathematics, Economics, History, and Psychology. Suppose that History and Psychology are taught in large lectures of 200, but all the other classes are taught in small sections of 20 students each. What is the average class size?

Course	Number of students in each section	Number of Sections	Total number of students in this course
English	20	10	200
Mathematics	20	10	200
Economics	20	10	200
History	200	1	200
Psychology	200	1	200
Totals		32	1000

The average class size is $\frac{1000}{32} = 31.25$. This is the figure an administrator would report. It is also the figure that the faculty experience when reporting their teaching loads, since each class is reported by one instructor. It accurately describes the experience of the university in reporting how large, on the average, are our classes. However, it is a serious error to presume that a typical student experiences the same "average." In our example we have 200 students all having the same class size experience. Each one of them has five classes, three with 20 students and two with 200. Thus, they all compute their personal average class size as $\frac{460}{5} = 92$. Observe that every single student in our example experiences an average class size nearly three times as large as the university claims! How can this be? How can every student have classes that are so much larger than average? Shades of Lake Wobegon, where the children are all above average!

The answer is that when the institution computes average class size it counts each class exactly once, but when students compute their own personal average class size the large lectures get experienced and reported 200 times while each small class gets reported only 20 times. This shifts the average considerably toward the higher end. Neither the university nor the student is wrong or deceitful, but the averages they report are very different animals. The average experience of the faculty and the administration is truthfully represented by the university average; the typical student-experienced class size is not. In all fairness to our students, we must understand this difference, and we must avoid the temptation to compute the average class size in the customary manner and then shape our opinions and policies on the premise that this average will somehow reflect the typical student experience.

My example is highly contrived to keep the arithmetic simple, but I can prove mathematically that the student-experienced average is always greater than the institution average. How much greater depends upon the disparity between the largest and the smallest classes. If classes are nearly uniform, the difference is insignificant, perhaps only a tenth of a point. But when there is a tenfold disparity, the student experience can be more than double, as my example illustrates. Of course in the real world we don't have all students taking precisely the same courses, but the effects are similar. If you don't trust my selection of an example, construct your own. I guarantee a similar discrepancy, though possibly less severe.

Before giving a rigorous comparison, we need to specify how we compute the average class size experienced by students. This is not a trivial issue. Consider two contrived cases.

Case 1. Ann is enrolled in one class that has 20 students, while Bob has a schedule of five classes, each with 200 students. Clearly Ann's average is 20 and Bob's is 200. Together they average 110.

Case 2. Carl has a single class of size 200, while Diane is in five classes of size 20. Again, the average size for both students is 110.

Something doesn't seem quite right here. We get the same average in each case, but doesn't Case 1 suggest mostly large classes while Case 2 has mostly small? The problem is that we are treating each student's experience as having equal weight when we compute these averages. But the one-class experience of Ann and Carl shouldn't count as heavily as the 5-course experience of Bob and Diane. And if we have a student who has taken 40 classes, that experience should influence the average more strongly than any of these. To reflect this observation, we choose to use a weighted average by weighting each student's personal average by the number of courses he or she has taken. Thus in Case 1 we get $\frac{1 \cdot 20 + 5 \cdot 200}{1+5} = 170$, while Case 2 produces $\frac{1 \cdot 200 + 5 \cdot 20}{1+5} = 50$. In effect, the personal average disappears from the formula and is replaced by a sum of all the student class sizes observed, then divided by the total number of classes. The weighted average seems to do a better job of reflecting the reality of student experience.

We are now ready to introduce a bit of notation to allow a rigorous comparison. Assume that the number of classes (or sections) of size i is given by c_i, and the maximum class size is m. Notice that each size i appears c_i times, and the total number of classes is the sum of the c_i's. The university average class size $\bar{c}$ is therefore

$$\bar{c} = \frac{\sum_{i=1}^{m} i \cdot c_i}{\sum_{i=1}^{m} c_i}.$$

To compute the student-experienced average, observe that each class of size i gets reported by i students, for a total of ic_i reports. This makes the weighted average of all the student computed average class sizes

$$\bar{s} = \frac{\sum_{i=1}^{m} i \cdot ic_i}{\sum_{i=1}^{m} ic_i}.$$

In the student average $\bar{s}$ the larger classes have higher weights. The university average $\bar{c}$ weighs each class equally. Consequently, for any distribution of class sizes whatsoever, the student average will always exceed the university average. The only way for equality to occur is to have all classes with equal sizes.

If that explanation is not to your liking, we can use statistics. Since the variance of $\bar{c}$ is the average of the squares of the deviations from the mean, it is certainly nonnegative [**2**, p. 98], so in our problem we find

$$\text{Var}(\bar{c}) = \frac{\sum_{i=1}^{m} i^2 c_i}{\sum_{i=1}^{m} c_i} - (\bar{c})^2 = \frac{\sum_{i=1}^{m} i^2 c_i}{\sum_{i=1}^{m} c_i} - \left[\frac{\sum_{i=1}^{m} i \cdot c_i}{\sum_{i=1}^{m} c_i}\right]^2 \geq 0$$

Upon dividing by $\bar{c}$ we find

$$\left[\frac{\sum_{i=1}^{m} i^2 c_i}{\sum_{i=1}^{m} c_i}\right]\left[\frac{\sum_{i=1}^{m} c_i}{\sum_{i=1}^{m} i \cdot c_i}\right] - \left[\frac{\sum_{i=1}^{m} i \cdot c_i}{\sum_{i=1}^{m} c_i}\right] \geq 0$$

Canceling the $\sum c_i$ in the first term yields $\bar{s} - \bar{c} \geq 0$. So certainly the student-experienced weighted average is always greater than the university average, with equality only if the variance is zero; that is, only if all classes are precisely the same size.

Finally, if statistics is not your cup of tea, perhaps you are a fan of Cauchy and Schwarz. Define two vectors of dimension m, the maximum class size:

$$\vec{v} = [\sqrt{c_1}, \sqrt{c_2}, \sqrt{c_3}, \ldots, \sqrt{c_m}]$$
$$\vec{w} = [1\sqrt{c_1}, 2\sqrt{c_2}, 3\sqrt{c_3}, \ldots, m\sqrt{c_m}]$$

According to the Cauchy-Schwarz inequality [**1**, p. 195],

$$(\vec{v} \cdot \vec{v})(\vec{w} \cdot \vec{w}) \geq (\vec{v} \cdot \vec{w})^2$$
$$\left[\sum_{i=1}^{m} c_i\right]\left[\sum_{i=1}^{m} i^2 c_i\right] \geq \left[\sum_{i=1}^{m} i \cdot c_i\right]^2.$$

Upon dividing by $\left[\sum_{i=1}^{m} c_i\right]\left[\sum_{i=1}^{m} i \cdot c_i\right]$, we get, yet again, $\bar{s} \geq \bar{c}$. And when might these be equal? Why only if one vectors is a scalar multiple of the other. This happens only if precisely one term is nonzero. That is, if all classes are the same size.

In the real world this never happens. Consequently, the student average $\bar{s}$ *always exceeds* $\bar{c}$, the university reported average. Sometimes this can be by a small amount; sometimes, as in my hypothetical example, the effect is large. At a typical university I would guess that 90% of the students experience an average class size larger than what the university reports.

References

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Exhaustive sampling and related binomial identities

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There are many situations that involve repeated sampling from the same set of observations. For example, suppose a professor has a test bank of 100 questions for a particular course and randomly chooses 25 of these questions for the final exam each semester. A persistent but not very talented student repeats the course several times. Obviously, the student has no chance of having seen all the questions before taking the course four times. What is the probability that the student will have seen all the questions after k repetitions? That is, what is the probability that the entire test bank will have been exhausted after k repetitions?

A more practical example involves drug testing. Suppose, for example that a bicycle race has 100 contestants and consists of several stages where random samples of 20 contestants are taken at each stage and screened for banned substances. If the race has 10 stages, what is the probability that each contestant will be tested for banned substances at least once?

Probability of exhaustion. We will assume that we are selecting k samples of size n from a population containing N members. We want to find the probability of the event E that the population has been exhausted in the k samples. That is, every member of the population has been included in at least one sample. Denote the members of the population by $x_1, x_2, \dots, x_N$. We will calculate the probability of the complementary event E^C. Let E_i be the event that x_i has not been included in any of the k samples. Then $E^C = E_1 \cup E_2 \cup \dots \cup E_N$. By the addition law of probability and the method of inclusion and exclusion,

$$P(E^C) = \sum_{i=1}^{N} P(E_i) - \sum_{i<j} P(E_i \cap E_j) + \sum_{p<q<r} P(E_p \cap E_q \cap E_r) - \cdots, \quad (1)$$

where the last sum consists of all terms with $N - n$ intersections. That is because each sample has n distinct elements and so the greatest number of elements that cannot be included is $N - n$. For a particular x_i, let B_{ij} be the event that x_i is not included in the jth sample. Then $E_i = B_{i1} \cap B_{i2} \cap \cdots \cap B_{ik}$. Moreover,

$$P(B_{ij}) = \frac{\binom{N-1}{n}}{\binom{N}{n}}$$

since we must choose a sample from N elements without including x_i. Since the samples are chosen with replacement, the events $B_{i1}, B_{i2}, \dots, B_{ik}$ are independent, and so

$$P(E_i) = P(B_{i1} \cap B_{i2} \cap \cdots \cap B_{ik}) = \prod_{j=1}^{k} P(B_{ij}) = \prod_{j=1}^{k} \frac{\binom{N-1}{n}}{\binom{N}{n}} = \left[\frac{\binom{N-1}{n}}{\binom{N}{n}}\right]^k. \qquad (2)$$

Consider the first term in (1). There are $\binom{N}{1}$ ways to choose i, and each $P(E_i)$ has the probability given in (2), so this first term has the value

$$\binom{N}{1}\left[\frac{\binom{N-1}{n}}{\binom{N}{n}}\right]^k.$$

Next, consider the second term in (1). There are $\binom{N}{2}$ ways to choose the integers i and j between 1 and N with $i < j$. Also $E_i \cap E_j$ means that neither x_i nor x_j is in any of the k independent samples. But the probability that neither x_i nor x_j is in any of these independent samples is $\binom{N-2}{n}/\binom{N}{n}$. Since the samples are independent, the probability that neither x_i nor x_j is in any of the k independent samples is $\left[\binom{N-2}{n}/\binom{N}{n}\right]^k$. Consequently, the second term of the sum (1) is $\binom{N}{2}\left[\binom{N-2}{n}/\binom{N}{n}\right]^k$. Each of the later terms can be analyzed in a similar manner, and therefore, (1) can be rewritten as

$$P(E^C) = \binom{N}{1}\left[\frac{\binom{N-1}{n}}{\binom{N}{n}}\right]^k - \binom{N}{2}\left[\frac{\binom{N-2}{n}}{\binom{N}{n}}\right]^k + \binom{N}{3}\left[\frac{\binom{N-3}{n}}{\binom{N}{n}}\right]^k - \cdots$$
$$+ (-1)^{N-n+1}\binom{N}{N-n}\left[\frac{\binom{N-(N-n)}{n}}{\binom{N}{n}}\right]^k.$$

Simplifying, we get

$$P(E^C) = \frac{\binom{N}{1}\binom{N-1}{n}^k - \binom{N}{2}\binom{N-2}{n}^k + \binom{N}{3}\binom{N-3}{n}^k - \cdots + (-1)^{N-n+1}\binom{N}{N-n}\binom{n}{n}^k}{\binom{N}{n}^k}. \qquad (3)$$

Then the probability of exhaustion is $P(E) = 1 - P(E^C)$.

We again consider the example where a professor has a test bank of 100 questions and randomly chooses 25 for the final exam each semester and a persistent student continues to repeat the course each semester. What is the probability of the event E that the student has seen all 100 questions after taking the course k times? As we noted earlier, the student must take the course at least four times to have any chance of having seen all the questions. However, the probability of having seen them all in just four repetitions is only

$$P(E) = \frac{\binom{75}{25}\binom{50}{25}\binom{25}{25}}{\binom{100}{25}^3} = 4.66 \times 10^{-37},$$

so it is extremely unlikely that this will occur. It is not hard to write a program to calculate $P(E)$ but care must be taken to avoid overflow and underflow errors due to the nature of the numbers involved. The following table gives the results up to $k = 30$

when $N = 100$ and $n = 25$ (the numbers have been rounded off to eight decimal places). From this table, we see that students must be *very* persistent if they want at least a fifty-fifty chance of seeing all of the questions.

k	$P(E)$	k	$P(E)$
4	0.00000000	18	0.56269617
5	0.00000000	19	0.65090382
6	0.00000000	20	0.72545763
7	0.00000001	21	0.78656843
8	0.00000294	22	0.83553496
9	0.00011925	23	0.87411444
10	0.00150604	24	0.90413151
11	0.00890754	25	0.92726961
12	0.03159226	26	0.94498164
13	0.07868290	27	0.95846997
14	0.15278444	28	0.96870217
15	0.24836613	29	0.97644193
16	0.35514273	30	0.98228379
17	0.46257357		

Associated binomial identities. If $kn < N$, then it is impossible to have sampled all N members of the population with k samples of size n. Consequently, $P(E^C) = 1$ and the numerator of (3) must equal the denominator. This yields the following family of identities:

$$\binom{N}{1}\binom{N-1}{n}^k - \binom{N}{2}\binom{N-2}{n}^k + \binom{N}{3}\binom{N-3}{n}^k - \cdots + (-1)^{N-n+1}\binom{N}{N-n}\binom{n}{n}^k = \binom{N}{n}^k.$$

Rewritten with summation notation, this is

$$\sum_{j=1}^{N-n} (-1)^{j+1}\binom{N}{j}\binom{N-j}{n}^k = \binom{N}{n}^k. \tag{4}$$

These identities hold for any positive integer k for which $kn < N$. As an example, we consider the case $N = 7$ and $n = 2$. Part of Pascal's triangle is given below, and the numbers involved in the identities are in boldface type. In this case, (4) holds for $k = 1$, 2, and 3.

	n							
N	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2	**1**					
3	1	3	**3**	1				
4	1	4	**6**	4	1			
5	1	5	**10**	10	5	1		
6	1	6	**15**	20	15	6	1	
7	1	**7**	**21**	**35**	**35**	**21**	7	1

Here $\binom{N}{n} = 21$. The first factors of the terms in the sum on the left side of (4) start with $\binom{N}{1} = 7$ and progress to the right along the row for $N = 7$. The second factors of these terms are raised to the kth power and begin with $\binom{N-1}{2} = 15$ and progress up the column for $n = 2$. The numerical values for the identities in this example for $k = 1, 2,$ and 3 are given below.

$$7 \cdot 15 - 21 \cdot 10 + 35 \cdot 6 - 35 \cdot 3 + 21 \cdot 1 = 21$$
$$7 \cdot 15^2 - 21 \cdot 10^2 + 35 \cdot 6^2 - 35 \cdot 3^2 + 21 \cdot 1^2 = 21^2$$
$$7 \cdot 15^3 - 21 \cdot 10^3 + 35 \cdot 6^3 - 35 \cdot 3^3 + 21 \cdot 1^3 = 21^3$$

The number of identities in the family is determined by how small n is relative to N. For example, if $N = 30$ and $n = 4$, then (4) holds for $k \leq 7$. The general relationship of each identity with respect to Pascal's triangle is the same as in the example. The first and second factors for the terms on the left side of (4) are found by starting with $\binom{N}{1}$ and moving to the right for the first factor and starting with $\binom{N-1}{2}$ and moving upwards for the second factor. The extensive literature on binomial coefficients has identities similar to the case where $k = 1$:

$$\sum_{j=1}^{N-n} (-1)^{j+1} \binom{N}{j} \binom{N-j}{n} = \binom{N}{n}.$$

For example, the reader is referred to the first chapter of Riordan's classic book, *Combinatorial Identities*. However, these identities do not have terms where factors are raised to an arbitrary power k as is the case in (4). The identity (4) is interesting in that it holds for all positive integers less than N/n. This allows us to write identities that hold for any number of consecutive integers but not beyond. For example, if $N = 1000$ and $n = 10$, then (4) holds for all $k \leq 99$ but not for any values beyond 99.

References

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Controlling the discrepancy in marginal analysis calculations

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Despite technology, we professors still love our little tricks in designing shortcuts and problems involving "nice numbers" that lead to easily predictable outcomes. Here's one such shortcut that I discovered recently.

Consider the typical case of a quadratic cumulative-cost function, often encountered in Calculus I and "Business Calculus" as a differentiation application. Here a hypothetical business produces x "widgets" for a total cost of $C(x) = ax^2 + bx + c$. (In order for $C(x)$ to be increasing, we restrict our attention to $0 \leq x \leq -b/2a$ with

$a < 0, b > 0$, and $c > 0$.) At production level x, the cost of the next item (item number x + 1) is $C(x+1) - C(x)$. This is the *marginal cost* of that item, sometimes called the marginal cost at level x. Inasmuch as

$$C(x+1) - C(x) = \frac{C(x+1) - C(x)}{1} \approx C'(x),$$

textbooks often define the marginal cost to be $C'(x)$ instead of that being an approximation. Regardless of which is definition and which is theorem, how do they differ for a quadratic function?

Easy!

$$\begin{aligned}
&\left|[C(x+1) - C(x)] - C'(x)\right| \\
&\quad = \left|a(x+1)^2 + b(x+1) + c - (ax^2 + bx + c) - (2ax + b)\right| \\
&\quad = |a|,
\end{aligned}$$

which gives us a nice simple result. (In the linear case, when $a = 0$, the two expressions of course give the same marginal cost at every production level.) So, if you want your two computations of marginal cost ($C(x+1) - C(x)$ and $C'(x)$) to be within 2 cents of one another, simply choose a quadratic model with $a = -0.02$. For example, if $C(x) = -0.02x^2 + 100x + 100000$ represents the cumulative cost of producing x DVD-players, then at any permissible production level x, the results of calculating marginal cost by the two methods will always differ by precisely 2 cents.

In closing, we note that this observation can be tied to other ideas in calculus and differential equations, such as the Mean Value Theorem. For instance, consider the question of which functions f satisfy

$$\frac{f(b) - f(a)}{b - a} = f'\left(\frac{a+b}{2}\right)$$

for arbitrary a and b; that is, the point of attainment of the theorem is always the midpoint of the interval. The answer is the set of all linear and quadratic polynomials. Of course, we don't usually teach the Mean Value Theorem in business calculus courses, not even for quadratic functions. Nevertheless, instructors might welcome the following additional observation: For a quadratic function $C(x)$, $C(x+1) - C(x) = C'(x + \frac{1}{2})$. This gives still another way of thinking about the discrepancy, namely, as $\left|C'(x + \frac{1}{2}) - C'(x)\right|$.

Stirling's formula via Riemann sums

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Introduction. Recently Rzadkowski [**5**] used Riemann sums in a novel way to prove Stirling's famous asymptotic approximation to the factorial

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n,$$

that is,

$$\lim_{n\to\infty} \frac{n!}{\sqrt{n}} \left(\frac{e}{n}\right)^n = \sqrt{2\pi}, \tag{1}$$

without resort to Wallis's product formula (see the remark at the end of this paper) or the fact that

$$\log\left[\frac{n!}{\sqrt{n}}\left(\frac{e}{n}\right)^n\right] \text{ decreases to a positive limit } S.$$

Traditionally the latter is proved first and then Wallis's formula is used to determine the value of S. The literature abounds with proofs of varying degrees of sophistication (and different estimates for the speed of convergence); some involve a complex variable z in the role of n and $\Gamma(z+1)$ in place of $n!$. (See, for example [**7**, pp. 253–254, 465–469]). The historical significance of (1) is its role in deMoivre's proof of his Central Limit Theorem (CLT) only a few years after Stirling's discovery. Interestingly, CLT can nowadays be proved without (1), and (1) can then be deduced from it by probabilistic arguments; see [**1, 8**].

The aim here is to exploit Riemann sums "to the limit," and the mean value theorem (repeatedly!) in order to derive (1) with minimal means, suitable for classroom presentation or as a guided sequence of exercises.

The actual knowledge being presupposed is: Continuous functions have maxima and minima and are uniformly continuous on closed, bounded intervals, Cauchy's convergence criterion, the mean value theorem (MVT), the chain rule, continuity and differentiability of the functions sine, exponential (on the real line) and logarithm (on the positive reals). The notation $e^{i\theta}$ is employed a few times, but only as shorthand. The notation "$a := \ldots$" signals the reader that this is the *definition* of a; and our notation for intervals in $\mathbb{R}$ is that of Bourbaki, to avoid confusion with ordered pairs.

Step 1. A weaker relation than (1) that is needed for its proof is

$$\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e. \tag{2}$$

Here is a short path to this using only the above prerequisites. It too could be broken out into two or three little exercises. Consider any sequence of real numbers b_n that converges to a real limit b. Given $\varepsilon > 0$, first pick N so that $|b - b_n| \le \varepsilon$ for all $n \ge N$. Then for all $n \ge N + (|b - b_1| + \ldots + |b - b_N|)/\varepsilon$,

$$\left| b - \frac{1}{n}\sum_{k=1}^{n} b_k \right| = \left| \frac{1}{n}\sum_{k=1}^{n}(b - b_k)\right| \le \frac{1}{n}\sum_{k=1}^{N}|b - b_k| + \frac{1}{n}\sum_{k=N+1}^{n}|b - b_k|$$
$$\le \frac{n-N}{n}(\varepsilon + \varepsilon) < 2\varepsilon,$$

showing that

$$\lim_{n\to\infty} b_n = b \Longrightarrow \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} b_k = b, \tag{CC}$$

a fact associated with the names Cauchy and Cesàro. Another familiar result of Cauchy's follows:

$$\text{If } a_n \text{ and } L \text{ are positive numbers and } \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = L, \text{ then } \lim_{n\to\infty} a_n^{1/n} = L. \quad (3)$$

For the hypothesis of (3) and continuity of the logarithm tell us that (with $a_0 := 1$) the numbers $b_n := \log(a_n/a_{n-1})$ converge to $b := \log L$, so according to (CC)

$$\frac{1}{n}\log a_n = \frac{1}{n}\sum_{k=1}^{n} b_k \to b.$$

Then by continuity of the exponential

$$a_n^{1/n} = e^{(1/n)\log a_n} \to e^b = L.$$

For $a_n := n^n/n!$, we have

$$\log(a_{n+1}/a_n) = n\log(1+1/n) = \frac{\log(1+1/n) - \log(1)}{1/n} \to \log'(1) = 1,$$

whence $a_{n+1}/a_n \to e$, showing that (2) is a consequence of (3). Another consequence of (3), useful in the sequel, gotten by taking $a_n := n$, is the elementary fact

$$\lim_{n\to\infty} \sqrt[n]{n} = 1. \quad (4)$$

Mean values and generalized averages. Let $\mathbb{N}$ denote the positive integers, $\mathbb{R}$ the reals. For $f : [0, 1] \to \mathbb{R}$ and $n \in \mathbb{N}$, define

$$R_n(f) := \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \quad (5)$$

and

$$I(f) := \lim_{n\to\infty} R_n(f), \quad (6)$$

whenever this limit exists in $\mathbb{R}$, calling it the *mean value* of f. The notation is meant to suggest Riemann sums and integrals, because if f is Riemann integrable, the limit does exist and it is the Riemann integral of f. The present exposition does not presuppose the reader knows this fact, but some of the proof techniques below are those usually employed at the beginning of a rigorous development of Riemann integration. Teachers and other readers with that knowledge can skip Step 2.

Step 2. If f is continuous on [0, 1], then limit (6) exists.

Proof. Given $\varepsilon > 0$, use uniform continuity of f to select $N = N_\varepsilon \in \mathbb{N}$ so that

$$x, t \in [0, 1] \text{ and } |x - t| \le \frac{1}{N} \Rightarrow |f(x) - f(t)| \le \varepsilon. \quad (7)$$

By organizing terms into blocks, we see that for any $n, m \in \mathbb{N}$,

$$R_{mn}(f) = \frac{1}{mn} \sum_{k=0}^{n-1} \sum_{r=1}^{m} f\left(\frac{km+r}{mn}\right),$$

whence

$$R_{mn}(f) - R_n(f) = \frac{1}{n} \sum_{k=0}^{n-1} \left\{ \frac{1}{m} \sum_{r=1}^{m} \left[f\left(\frac{k}{n} + \frac{r}{mn}\right) - f\left(\frac{k}{n} + \frac{m}{mn}\right) \right] \right\}.$$

Using (7) on each bracketed summand, we infer that $|R_{mn}(f) - R_n(f)| \leq \varepsilon$ whenever $m, n \geq N$. By symmetry, $|R_{mn}(f) - R_m(f)| \leq \varepsilon$ as well. Hence, $|R_n(f) - R_m(f)| \leq 2\varepsilon$, for all $m, n \geq N$, proving that $(R_n(f))_{n\in\mathbb{N}}$ is a Cauchy sequence. ■

Henceforth consider only continuous f. Its uniform continuity on [0, 1] can be further exploited to broaden the averaging process. Suppose that for each $n \in \mathbb{N}$ points

$$x_{k,n} \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$$

are given, $k = 1, \ldots, n$. Set

$$L_{k,n} := \text{minimum of } f \text{ on } \left[\frac{k-1}{n}, \frac{k}{n}\right], \quad M_{k,n} := \text{ maximum of } f \text{ on } \left[\frac{k-1}{n}, \frac{k}{n}\right].$$

Step 3.

$$S_n(f) := \frac{1}{n} \sum_{k=1}^{n} f(x_{k,n}) \text{ converges to } I(f) \text{ as } n \to \infty.$$

Proof. According to (7), $|f(x_{k,n}) - f(k/n)| \leq \varepsilon$ for all $k \in \{1, \ldots, n\}$ if $n \geq N_\varepsilon$, so $|R_n(f) - S_n(f)| \leq \varepsilon$ for such n.

Since $L_{k,n}$ and $M_{k,n}$ are realized by appropriate choices of $x_{k,n}$, it follows that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} L_{k,n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} M_{k,n} = I(f). \tag{8}$$

■

One form of the Fundamental Theorem of Calculus is an immediate consequence:

Step 4. If f is differentiable on [0, 1] (with finite one-sided derivatives at endpoints) and f' is continuous on [0, 1], then

$$I(f') = f(1) - f(0).$$

Proof. By the MVT there are

$$x_{k,n} \in \left[\frac{k-1}{n}, \frac{k}{n}\right] \quad \text{such that} \quad \frac{1}{n} f'\left(x_{k,n}\right) = f\left(\frac{k}{n}\right) - f\left(\frac{k-1}{n}\right).$$

The sum $S_n(f')$ based on these points evidently equals $f(1) - f(0)$, for every n. ■

A slight modification of the proof of Step 2 leads to a very useful quantitative refinement of it [**5**], namely

Step 5. If f is differentiable on [0, 1] and f' is continuous there, then

$$\lim_{n\to\infty} n[R_n(f) - I(f)] = \frac{f(1) - f(0)}{2}.$$

Proof. Consider $n, m \in \mathbb{N}$, $k \in \{0, 1, \ldots, n-1\}$, $r \in \{1, 2, \ldots, m\}$, and set

$$M'_{k,n} := \text{maximum of } f' \text{ on } \left[\frac{k}{n}, \frac{k+1}{n}\right].$$

Thanks to the MVT,

$$f\left(\frac{k}{n} + \frac{m}{mn}\right) - f\left(\frac{k}{n} + \frac{r}{mn}\right) \le \frac{m-r}{mn} M'_{k,n}.$$

Therefore

$$\begin{aligned} n\,[R_n(f) - R_{mn}(f)] &= \sum_{k=0}^{n-1} \left\{ \frac{1}{m} \sum_{r=1}^{m} \left[f\left(\frac{k}{n} + \frac{m}{mn}\right) - f\left(\frac{k}{n} + \frac{r}{mn}\right) \right] \right\} \\ &\le \sum_{k=0}^{n-1} \left\{ \frac{1}{m} \sum_{r=1}^{m} \frac{m-r}{mn} M'_{k,n} \right\} = \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{m^2} M'_{k,n} \sum_{r=1}^{m} (m-r) \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{2}\left(1 - \frac{1}{m}\right) M'_{k,n}. \end{aligned}$$

Letting $m \to \infty$, we see that

$$n[R_n(f) - I(f)] \le \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{2} M'_{k,n}.$$

Using Step 3, in the version (8), and the result of Step 4, we find that for each $\varepsilon > 0$,

$$n[R_n(f) - I(f)] < \frac{1}{2} I(f') + \varepsilon = \frac{f(1) - f(0)}{2} + \varepsilon$$

holds whenever n is sufficiently large. Similar reasoning with $L'_{k,n} :=$ minimum of f' on $\left[\frac{k}{n}, \frac{k+1}{n}\right]$ shows that the left-hand side exceeds $\frac{1}{2}[f(1) - f(0)] - \varepsilon$ for all sufficiently large n.

■

The function sinc. Sinc is the customary designation for the function s defined by

$$s(x) := \begin{cases} \frac{1}{x} \sin x, & x \in \mathbb{R}\backslash\{0\} \\ 1, & x = 0. \end{cases} \tag{9}$$

Step 6. The function s is an even function which is continuously differentiable on $\mathbb{R}$ and strictly positive in the open interval $]-\pi, \pi[$.

Proof. Only the behavior at 0 is in question. Since $\lim_{x\to 0} s(x) = \sin'(0) = 1$, s is continuous at 0. For $u \in]0, \pi[$ the MVT furnishes $y \in]0, u[$ such that

$$(\cos u + u^2/2) - 1 = u[\cos'(y) + y] = u[-\sin y + y] > 0.$$

Also for $x \in]0, \pi[$, the MVT furnishes $u \in]0, x[$ such that

$$\sin x - x + \frac{x^3}{6} = x\left[\sin'(u) - 1 + \frac{u^2}{2}\right] = x\left[\cos u - 1 + \frac{u^2}{2}\right],$$

which we just saw is positive. That is,

$$0 \le x - \sin x < \frac{x^3}{6},$$

whence (for $x \ne 0$),

$$\frac{1}{x}\left|\frac{\sin x}{x} - 1\right| < \frac{x}{6}.$$

It follows that $s'(0)$ exists and equals 0. Since, for $x \ne 0$,

$$s'(x) = \frac{x\cos x - \sin x}{x^2} = \frac{\cos x - 1}{x} - \frac{s(x) - s(0)}{x},$$

we see that $\lim_{x\to 0} s'(x) = \cos'(0) - s'(0) = 0 = s'(0)$. Hence s' is continuous at 0. ■

Step 7. With s as defined in (9), the function

$$f(x) := \log\left(s\left(\frac{\pi}{2}x\right)\right), \quad x \in]-2, 2[\tag{10}$$

is continuously differentiable and satisfies

$$\lim_{n\to\infty} n[R_n(f) - I(f)] = \log\sqrt{2/\pi}. \tag{11}$$

Proof. The continuous differentiability follows from Step 6 and the chain rule, and (11) from Step 5 and definitions (10) and (9). ■

A trigonometric identity. The next goal is the (recondite?) identity (13), for which Rzadkowski [**5**] suggests a short proof that incidentally involves complex numbers. For a result like (1) that's wholly real, a wholly real proof might seem desirable (despite Hadamard's famous observation on his proof of the prime number theorem), and Rzadkowski [**6**] provides a few such proofs.

Recall that if $z_1, z_2, \ldots, z_m$ are distinct complex numbers and are the zeros of the mth-degree monic polynomial $P(z)$, then $P(z) = \prod_{j=1}^{m}(z - z_j)$.

Step 8. For any integer $n > 1$, the $2n$ numbers $e^{2\pi i(j/2n)}$, for $-n < j \le n$ are the $(2n)$th-roots of 1. Consequently, since $j = 0$ and $j = n$ correspond to the roots 1 and -1, respectively,

$$z^{2n} - 1 = \prod_{-n<j\le n}(z - e^{2\pi ij/2n}) = (z-1)(z+1)\prod_{0<|j|<n}(z - e^{\pi ij/n})$$

$$= (z^2 - 1) \prod_{0<k<n} (z - e^{\pi i k/n})(z - e^{-\pi i k/n})$$

$$= (z^2 - 1) \prod_{0<k<n} \left(z^2 - 2z \cos \frac{\pi k}{n} + 1\right).$$

Hence,

$$\frac{z^{2n} - 1}{z^2 - 1} = \prod_{k=1}^{n-1} (z^2 - 2z \cos \frac{\pi k}{n} + 1) \qquad \forall z \in \mathbb{R} \backslash \{\pm 1\}, n > 1. \tag{12}$$

The left side of (12) is a difference quotient for the nth-power function, and it converges to $n \cdot 1^{n-1} = n$ as $z \to 1$. Hence

$$n = \prod_{k=1}^{n-1} (1 - 2 \cos \frac{\pi k}{n} + 1) = \prod_{k=1}^{n-1} 4 \sin^2 \left(\frac{k\pi}{2n}\right),$$

whence

$$\frac{\sqrt{n}}{2^{n-1}} = \prod_{k=1}^{n-1} \sin \left(\frac{k\pi}{2n}\right) \qquad \forall n > 1.$$

On the right we may adjoin the factor corresponding to $k = n$, as it is 1, and then the equality holds for $n = 1$ as well. That is,

$$\frac{\sqrt{n}}{2^{n-1}} = \prod_{k=1}^{n} \sin \left(\frac{k\pi}{2n}\right) \qquad \forall n \in \mathbb{N}. \tag{13}$$

Final assault. The base camp fully provisioned by (2), (4), and (13), we can finish the ascent with

Step 9. The continuous function f in (10) satisfies

$$R_n(f) = \log \left(\frac{n}{\sqrt[n]{n!}}\right) - \log \pi + \frac{1}{2} \log(n^{1/n}) + \frac{1}{n} \log 2 \qquad \forall n \in \mathbb{N}, \tag{14}$$

and consequently

$$I(f) = 1 - \log \pi. \tag{15}$$

Proof. For $x \in]0, 1]$, $f(x) = \log\left(\sin\left(\frac{\pi}{2}x\right)\right) - \log\left(\frac{\pi}{2}x\right)$. Hence

$$\begin{aligned} nR_n(f) &= \sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \sum_{k=1}^{n} \log \sin \left(\frac{\pi}{2}\frac{k}{n}\right) - \sum_{k=1}^{n} \log \left(\frac{\pi}{2}\frac{k}{n}\right) \\ &= \log \prod_{k=1}^{n} \sin \left(\frac{\pi}{2}\frac{k}{n}\right) - \log \prod_{k=1}^{n} \left(\frac{\pi}{2}\frac{k}{n}\right) \\ &= \log \frac{\sqrt{n}}{2^{n-1}} - \log \left(\left(\frac{\pi}{2n}\right)^n n!\right) \quad \text{by (13)}, \end{aligned} \tag{16}$$

from which (14) follows via the homomorphic property of the logarithm. To get (15) from (14), use (6), (2), (4), and the continuity of log. ∎

Step 10. From (15) and (16) and a little computation, we get

$$n[R_n(f) - I(f)] - \log\sqrt{2/\pi} = \log\left(\frac{n^n\sqrt{2\pi n}}{n!e^n}\right) \qquad \forall n \in \mathbb{N}. \tag{17}$$

We let $n \to \infty$ in (17) and invoke (11) to learn that

$$\lim_{n\to\infty} \log\left(\frac{n^n\sqrt{2\pi n}}{n!e^n}\right) \quad \text{exists and equals 0,}$$

which, because of the continuity of the exponential function at 0, is equivalent to Stirling's formula (1).

Remark. Rzadkowski [**5**] points out that if this machine is run on a slightly different fuel, namely

$$f(x) := \log\left((1+x)s\left(\frac{\pi}{2}x\right)\right)$$

instead of (10), it outputs the beautiful product formula of Wallis:

$$\frac{\pi}{2} = \prod_{n=1}^{\infty}\left(\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\right) = \prod_{n=1}^{\infty}\left(\frac{4n^2}{4n^2-1}\right).$$

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