Maximum Overhang

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1. INTRODUCTION. How far can a stack of n identical blocks be made to hang over the edge of a table? The question has a long history and the answer was widely believed to be of order $\log n$. Recently, Paterson and Zwick constructed n-block stacks with overhangs of order $n^{1/3}$, exponentially better than previously thought possible. We show here that order $n^{1/3}$ is indeed best possible, resolving the long-standing overhang problem up to a constant factor.

This problem appears in physics and engineering textbooks from as early as the mid-19th century (see, e.g., [**15**], [**20**], [**13**]). The problem was apparently first brought to the attention of the mathematical community in 1923 when J. G. Coffin [**2**] posed it in the "Problems and Solutions" section of this MONTHLY; no solution was presented there. The problem recurred from time to time over subsequent years, e.g., [**17, 18, 19, 12, 6, 5, 7, 8, 1, 4, 9, 10**], achieving much added notoriety from its appearance in 1964 in Martin Gardner's "Mathematical Games" column of *Scientific American* [**7**] and in [**8**, Limits of Infinite Series, p. 167].

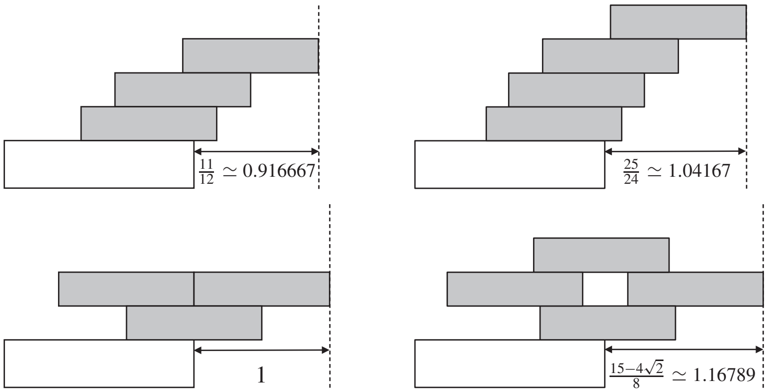

Figure 1. Optimal stacks with 3 and 4 blocks, compared to the corresponding harmonic stacks. The 4-block solution is from [**1**]. Like the harmonic stacks it can be made stable by minute displacements.

Most of the references mentioned above describe the now-classical *harmonic stacks* in which n unit-length blocks are placed one on top of the other, with the ith block from the top extending by $\frac{1}{2i}$ beyond the block below it. The overhang achieved by such stacks is $\frac{1}{2}H_n = \frac{1}{2}\sum_{i=1}^{n}\frac{1}{i} \sim \frac{1}{2}\ln n$. The cases $n = 3$ and $n = 4$ are illustrated at the top of Figure 1 above, and the cases $n = 20$ and $n = 30$ are shown in the background of Figure 2. Verifying that harmonic stacks are *balanced* and can be made *stable* (see definitions in the next section) by minute displacements is an easy exercise. (This is the form in which the problem appears in [**15**, pp. 140–141], [**20**, p. 183], and [**13**, p. 341].) Harmonic stacks show that arbitrarily large overhangs can be achieved if sufficiently many blocks are available. They have been used extensively as an introduction to recurrence relations, the harmonic series, and simple optimization problems (see, e.g., [**9**]).

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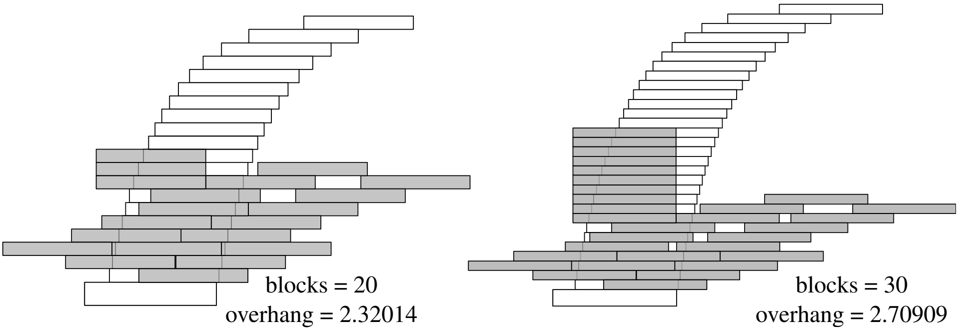

Figure 2. Optimal stacks with 20 and 30 blocks from [**14**] with corresponding harmonic stacks in the background.

Building a stack with an overhang is, of course, also a construction challenge in the real world. Figure 3 shows how the Mayans used such constructions in 900 BC in corbel arches. In contrast with a true arch, a corbel arch does not use a keystone, so the stacks forming the sides do not benefit from leaning against each other.

Figure 3. Mayan corbel arch from Cahal Pech, Belize, 900 BC (picture by Clark Anderson/Aquaimages under the Creative Commons License [**3**]).

1.1. How far can you go? Many readers of the above-mentioned references were led to believe that $\frac{1}{2}H_n$ ($\sim \frac{1}{2}\ln n$), the overhang achieved by harmonic stacks, is the *maximum* overhang that can be achieved using n blocks. This is indeed the case under the restriction, explicit or implicit in some of these references, that the blocks should be stacked in a *one-on-one* fashion, with at most one block resting on each block. It has been known for some time, however, that larger overhangs may be obtained if the one-on-one restriction is lifted. Three blocks, for example, can easily be used to obtain an overhang of 1. Ainley [**1**] found that four blocks can be used to obtained an overhang of about 1.16789, as shown at the bottom right of Figure 1, and this is more than 10% larger than the overhang of the corresponding harmonic stack. Using computers, Paterson and Zwick [**14**] found the optimal stacks with a given limited number of blocks. Their solutions with 20 and 30 blocks are shown in Figure 2.

Now what happens when n grows large? Can general stacks, not subject to the one-on-one restriction, improve upon the overhang achieved by the harmonic stacks by more than a constant factor, or is overhang of order $\log n$ the best that can be achieved? In a recent cover article in the *American Journal of Physics*, Hall [**10**] observes that the addition of counterbalancing blocks to one-on-one stacks can double (asymptotically) the overhang obtainable by harmonic stacks. However, he then incorrectly concludes that no further improvement is possible, thus perpetuating the order $\log n$ "mythology".

Recently, however, Paterson and Zwick [**14**] discovered that the modest improvements gained for small values of n by using layers with multiple blocks mushroom into an exponential improvement for large values of n, yielding an overhang of order $n^{1/3}$ instead of just $\log n$.

1.2. Can we go further? But is $n^{1/3}$ the right answer, or is it just the start of another mythology? In their deservedly popular book *Mad About Physics* [**11**, Challenge 271: A staircase to infinity, p. 246], Jargodzki and Potter rashly claim that inverted triangles (such as the one shown on the left of Figure 4) are balanced. If so, they would achieve overhangs of order $n^{1/2}$. It turns out, however, that already the 3-row inverted triangle is unbalanced, and collapses as shown on the right of Figure 4, as do all larger inverted triangles.

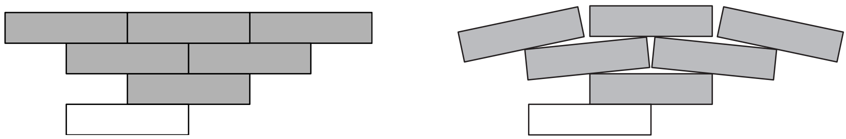

Figure 4. A 3-row inverted triangle is unbalanced.

The collapse of the 3-row triangle begins with the lifting of the middle block in the top row. It is tempting to try to avoid this failure by using a diamond shape instead, as illustrated in Figure 5. Diamonds were considered by Drummond [**4**] and, like the inverted triangle, they would achieve an overhang of order $n^{1/2}$, though with a smaller leading constant. The analysis of the balance of diamonds is slightly more complicated than that of inverted triangles, but it can be shown that d-diamonds, i.e., diamonds that have d blocks in their largest row, are balanced if and only if $d < 5$. In Figure 5 we see a practical demonstration with $d = 5$.

It is not hard to show that particular constructions like larger inverted triangles or diamonds are unbalanced. This imbalance of inverted triangles and diamonds was already noted in [**14**]. However, this does not rule out the possibility of a smarter balanced way of stacking n blocks so as to achieve an overhang of order $n^{1/2}$, and that would be much better than the above mentioned overhang of order $n^{1/3}$ achieved by Paterson and Zwick [**14**]. Paterson and Zwick did consider this general question (in the preliminary SODA'06 version). They did not rule out an overhang of order $n^{1/2}$, but they proved that no larger overhang would be possible. Thus their work shows that

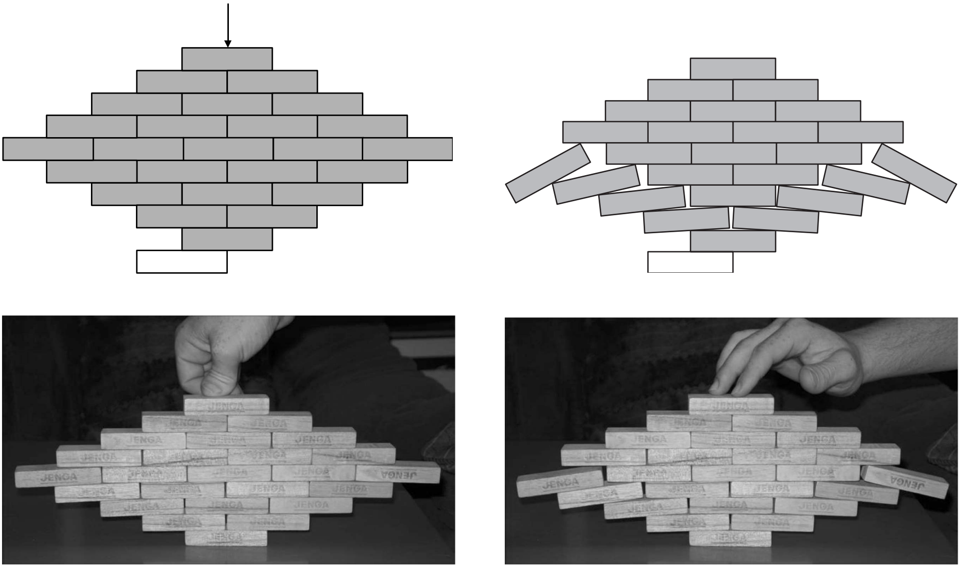

Figure 5. The instability of a 5-diamond in theory and practice.

the order of the maximum overhang with n blocks has to be somewhere between $n^{1/3}$ and $n^{1/2}$.

1.3. Our result. We show here that an overhang of order $n^{1/3}$, as obtained by [**14**], is in fact best possible. More specifically, we show that any n-block stack with an overhang of at least $6n^{1/3}$ is unbalanced, and hence must collapse. Thus we conclude that the maximum overhang with n blocks is of order $n^{1/3}$.

1.4. Contents. The rest of this paper is organized as follows. In the next section we present a precise mathematical definition of the overhang problem, explaining in particular when a stack of blocks is said to be *balanced* (and when it is said to be *stable*). In Section 3 we briefly review the Paterson-Zwick construction of stacks that achieve an overhang of order $n^{1/3}$. In Section 4 we introduce a class of abstract "mass movement" problems and explain the connection between these problems and the overhang problem. In Section 5 we obtain bounds for mass movement problems that imply the order $n^{1/3}$ upper bound on overhang. We end in Section 6 with some concluding remarks and open problems.

2. THE MODEL. We briefly state the mathematical definition of the overhang problem. For more details, see [**14**]. As in previous papers, e.g., [**10**], the overhang problem is taken here to be a two-dimensional problem: each block is represented by a frictionless rectangle whose long sides are parallel to the table. Our upper bounds apply, however, in much more general settings, as will be discussed in Section 6.

2.1. Stacks. Stacks are composed of blocks that are assumed to be identical, homogeneous, frictionless rectangles of unit length, unit weight, and height h. Our results here are clearly independent of h, and our figures use any convenient height. Previous authors have thought of blocks as cubes, books, coins, playing cards, etc.

A stack $\{B_1, \ldots, B_n\}$ of n blocks resting on a flat table is specified by giving the coordinates (x_i, y_i) of the lower left corner of each block B_i. We assume that the upper right corner of the table is at $(0, 0)$ and that the table extends arbitrarily far to the left.

Thus block B_i is identified with the box $[x_i, x_i + 1] \times [y_i, y_i + h]$ (its length aligned with the x-axis), and the table, which we conveniently denote by B_0, with the region $(-\infty, 0] \times (-\infty, 0]$. Two blocks are allowed to touch each other, but their interiors must be disjoint.

We say that block B_i *rests on* block B_j if $B_i \cap B_j \neq \emptyset$ and $y_i = y_j + h$. If $B_i \cap B_0 \neq \emptyset$ and $y_i = 0$, then B_i rests on the table. If B_i rests on B_j, we let $I_{ij} = B_i \cap B_j = [a_{ij}, b_{ij}] \times \{y_i\}$ be their *contact interval*. If $j \geq 1$, then $a_{ij} = \max\{x_i, x_j\}$ and $b_{ij} = \min\{x_i+1, x_j+1\}$. If $j = 0$ then $a_{i0} = x_i$ and $b_{i0} = \min\{x_i+1, 0\}$. (Note that we do allow single-point contact intervals here and so a block may rest on up to three other blocks; this convention only strengthens our impossibility results. In [**14**], mainly concerned with constructions and stability, a more conservative convention required proper contact intervals.)

The *overhang* of a stack is defined to be $\max_{i=1}^{n}(x_i+1)$.

2.2. Forces, equilibrium, and balance. Let $\{B_1, \dots, B_n\}$ be a stack composed of n blocks. If B_i rests on B_j, then B_j may apply an upward force of $f_{ij} \geq 0$ on B_i, in which case B_i will reciprocate by applying a downward force of the same magnitude on B_j. Since the blocks and table are frictionless, all the forces acting on them are vertical. The force f_{ij} may be assumed to be applied at a single point (x_{ij}, y_{ij}) in the contact interval I_{ij}. A downward gravitational force of unit magnitude is applied on B_i at its center of gravity $(x_i + 1/2, y_i + h/2)$.

Definition 2.1 (Equilibrium). Let B be a homogeneous block of unit length and unit weight, and let a be the x-coordinate of its left edge. Let $(x_1, f_1), (x_2, f_2), \dots, (x_k, f_k)$ be the positions and the magnitudes of the upward forces applied to B along its bottom edge, and let $(x'_1, f'_1), (x'_2, f'_2), \dots, (x'_{k'}, f'_{k'})$ be the positions and magnitudes of the upward forces applied by B, along its top edge, on other blocks of the stack. Then B is said to be in *equilibrium* under these collections of forces if and only if

$$\sum_{i=1}^{k} f_i = 1 + \sum_{i=1}^{k'} f'_i\,, \quad \sum_{i=1}^{k} x_i f_i = \left(a + \frac{1}{2}\right) + \sum_{i=1}^{k'} x'_i f'_i\,.$$

The first equation says that the *net force* applied to B is zero while the second says that the *net moment* is zero.

Definition 2.2 (Balance). A stack $\{B_1, \dots, B_n\}$ is said to be *balanced* if there exists a collection of forces acting between the blocks along their contact intervals such that, under this collection of forces and the gravitational forces acting on them, all blocks are in equilibrium.

The stacks presented in Figures 1 and 2 are balanced. They are, however, *precariously* balanced, with some minute displacement of their blocks leading to imbalance and collapse. A stack can be said to be *stable* if all stacks obtained by sufficiently small displacements of its blocks are balanced. We do not make this definition formal as it is not used in the rest of the paper, though we refer to it in some informal discussions.

A schematic description of a balanced stack and a collection of balancing forces acting between its blocks is given in Figure 6. Only upward forces are shown in the figure but corresponding downward forces are, of course, present. (We note in passing that balancing forces, when they exist, are in general not uniquely determined. This phenomenon is referred to as *static indeterminacy*.)

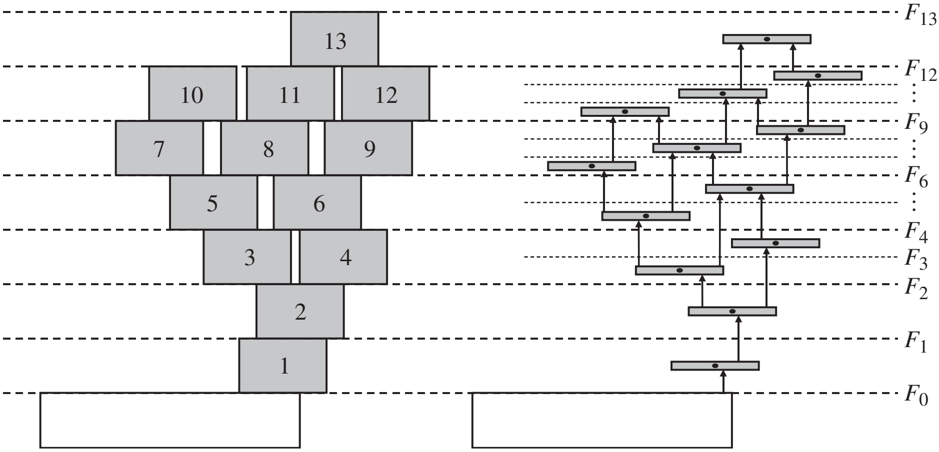

Figure 6. Balancing collections of forces within a stack.

We usually adopt the convention that the blocks of a balanced stack are numbered consecutively from bottom to top and, within each level, from left to right. Block B_1 is then the leftmost block in the lowest level while B_n is the rightmost block at the top level. For $0 \le i \le n$, we let F_i be a collection of upward balancing forces applied by blocks in $\{B_0, B_1, \dots, B_i\}$ on blocks in $\{B_{i+1}, \dots, B_n\}$ (see Figure 6).

Let us examine the relationship between two consecutive collections F_i and F_{i+1}. The only forces present in F_i but not in F_{i+1} are upward forces applied to B_i, while the only forces present in F_{i+1} but not in F_i are upward forces applied by B_i to blocks resting upon it. If we let $(x_1, f_1), (x_2, f_2), \dots, (x_k, f_k)$ be the positions and the magnitudes of the upward forces applied to B_i, and $(x'_1, f'_1), (x'_2, f'_2), \dots, (x'_{k'}, f'_{k'})$ be the positions and magnitudes of the upward forces applied by B_i, and if we let a be the x-coordinate of the left edge of B_i, we get by Definitions 2.1 and 2.2 that $\sum_{i=1}^{k} f_i = 1 + \sum_{i=1}^{k'} f'_i$ and $\sum_{i=1}^{k} x_i f_i = (a + \frac{1}{2}) + \sum_{i=1}^{k'} x'_i f'_i$. Block B_i thus *rearranges* the forces in the interval $[a, a+1]$ in a way that preserves the total magnitude of the forces and their total moment, when its own weight is taken into account. Note that all forces of F_0 act in nonpositive positions, and that if B_k is a rightmost block in a stack and the overhang achieved by it is d, then the total magnitude of the forces in F_{k-1} that act at or beyond position $d - 1$ should be at least 1. These simple observations play a central role in the rest of the paper.

2.3. The overhang problem. The natural formulation of the overhang problem is now:

What is the maximum overhang achieved by a *balanced* n-block stack?

The main result of this paper is:

Theorem 2.3. *The overhang achieved by a balanced n-block stack is at most* $6n^{1/3}$.

The fact that the stacks in the theorem above are required to be balanced, but not necessarily stable, only strengthens our result. By the nature of the overhang problem, stacks that achieve a maximum overhang are on the verge of collapse and thus unstable. In most cases, however, overhangs arbitrarily close to the maximum overhang may be obtained using stable stacks. (Probably the only counterexample is the case $n = 3$.)

3. THE PATERSON-ZWICK CONSTRUCTION. Paterson and Zwick [**14**] describe a family of balanced n-block stacks that achieve an overhang of about $(3n/16)^{1/3} \simeq 0.57n^{1/3}$. More precisely, they construct for every integer $d \geq 1$ a balanced stack containing $d(d-1)(2d-1)/3 + 1 \simeq 2d^3/3$ blocks that achieves an overhang of $d/2$. Their construction, for $d = 6$, is illustrated in Figure 7. The construction is an example of what they term a *brick-wall* stack, which resembles the simple "stretcher-bond" pattern in real-life bricklaying. In each row the blocks are contiguous, with each block centered over the ends of blocks in the row beneath. Overall the stack is symmetric and has a roughly parabolic shape, with vertical axis at the table edge.

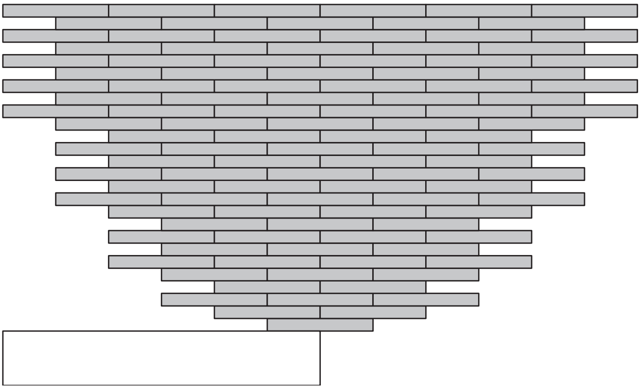

Figure 7. A "6-stack" consisting of 111 blocks and giving an overhang of 3, taken from [**14**].

The stacks of [**14**] are constructed in the following simple manner. A t-*row* is a row of t adjacent blocks, symmetrically placed with respect to $x = 0$. An r-*slab* has height $2r - 3$ and consists of alternating r-rows and $(r-1)$-rows, the bottom and top rows being r-rows. An r-slab therefore contains $r(r-1) + (r-1)(r-2) = 2(r-1)^2$ blocks. A 1-*stack* is a single block balanced at the edge of the table; a d-*stack* is defined recursively as the result of adding a d-slab symmetrically onto the top of a $(d-1)$-stack. The construction itself is just a d-stack and so has overhang $d/2$; its total number of blocks is given by $n = 1 + \sum_{r=1}^{d} 2(r-1)^2 = d(d-1)(2d-1)/3 + 1$. It is shown in [**14**], using an inductive argument, that d-stacks are balanced for any $d \geq 1$.

Why should a parabolic shape be appropriate? Some support for this comes from considering the effect of a block in spreading a single force of f acting from below into two forces of almost $f/2$ exerted upwards from its edges. This spreading behavior is analogous to a symmetric random walk on a line or to difference equations for the "heat-diffusion" process in a linear strip. In both cases we see that time of about d^2 is needed for effective spreading to width d, corresponding to a parabolic stack profile.

Our main result, Theorem 2.3, states that the parabolic stacks of [**14**] are optimal, up to constant factors. Better constant factors can probably be obtained, however. Paterson and Zwick [**14**] present some numerical evidence to suggest that, for large values of n, the overhang achievable using n blocks is at least $1.02n^{1/3}$. For more on this, see Section 6.

4. MASS MOVEMENT PROBLEMS. Our upper bound on the maximum achievable overhang is obtained by considering *mass movement* problems that are an abstraction of the way in which balancing forces "flow" though a stack of blocks. (See the discussion at the end of Section 2.2.) In a mass movement problem we are required to transform an initial *mass distribution* into a mass distribution that satisfies certain conditions. The key condition is that a specified amount of mass be moved to or beyond a certain position. We transform one mass distribution into another by performing local *moves* that redistribute mass within a unit interval in a way that preserves the total mass and the center of mass. Our goal is then to show that many moves are required to accomplish the task. As can be seen, masses here correspond to forces, mass distributions correspond to collections of forces, and moves mimic the effects of blocks.

The mass movement problems considered are formally defined in Sections 4.1 and 4.2. The correspondence between the mass movement problems considered and the overhang problem is established in Section 4.3. The bounds on mass movement problems that imply Theorem 2.3 are then proved in Section 5.

4.1. Distributions.

Definition 4.1 (Distributions and signed distributions). A discrete mass *distribution* is a set $\mu = \{(x_1, m_1), (x_2, m_2), \ldots, (x_k, m_k)\}$, where $k \geq 0$, $x_1, x_2, \ldots, x_k$ are real numbers, and $m_1, \ldots, m_k > 0$. A *signed distribution* μ is defined the same way, but without the requirement that $m_1, m_2, \ldots, m_k > 0$.

If $\mu = \{(x_1, m_1), (x_2, m_2), \ldots, (x_k, m_k)\}$ is a (signed) distribution, then for any set $A \subseteq \mathbb{R}$, we define

$$\mu(A) = \sum_{x_i \in A} m_i .$$

For brevity, we use $\mu(a)$ as a shorthand for $\mu(\{a\})$ and $\mu\{x > a\}$ as a shorthand for $\mu(\{x \mid x > a\})$. (Note that x here is a formal variable that does not represent a specific real number.) We similarly use $\mu\{x \geq a\}$, $\mu\{x < a\}$, $\mu\{a < x < b\}$, $\mu\{|x| \geq a\}$, etc., with the expected meaning.

We say that a (signed) distribution is *on* the interval $[a, b]$ if $\mu(x) = 0$, for every $x \notin [a, b]$.

For every $A \subseteq \mathbb{R}$, we let μ_A be the *restriction* of μ to A:

$$\mu_A = \{(x_i, m_i) \mid x_i \in A\}.$$

If μ_1 and μ_2 are two signed distributions, we let $\mu_1 + \mu_2$ and $\mu_1 - \mu_2$ be the signed distributions for which

$$(\mu_1 + \mu_2)(x) = \mu_1(x) + \mu_2(x), \qquad \text{for every } x \in \mathbb{R},$$
$$(\mu_1 - \mu_2)(x) = \mu_1(x) - \mu_2(x), \qquad \text{for every } x \in \mathbb{R}.$$

Definition 4.2 (Moments). Let $\mu = \{(x_1, m_1), (x_2, m_2), \ldots, (x_k, m_k)\}$ be a signed distribution and let $j \geq 0$ be an integer. The jth moment of μ is defined to be:

$$M_j[\mu] = \sum_{i=1}^{k} m_i x_i^j .$$

Note that $M_0[\mu]$ is the *total mass* of μ, $M_1[\mu]$ is the *torque* of μ, with respect to the origin, and $M_2[\mu]$ is the *moment of inertia* of μ, again with respect to the origin. If $M_0[\mu] \neq 0$, we let $C[\mu] = M_1[\mu]/M_0[\mu]$ be the *center of mass* of μ.

Less standard, but crucial for our analysis, is the following definition.

Definition 4.3 (Spread). The *spread* of a distribution $\mu = \{(x_1, m_1), (x_2, m_2), \ldots, (x_k, m_k)\}$ is defined as follows:

$$S[\mu] = \sum_{i<j} |x_i - x_j|\, m_i\, m_j.$$

If $M_0[\mu] = 1$, then μ defines a discrete random variable X for which $\Pr[X = x] = \mu(x)$, for every $x \in \mathbb{R}$. The spread $S[\mu]$ is then half the average distance between two independent drawings from μ. We also then have $M_1[\mu] = E[X]$ and $M_2[\mu] = E[X^2]$. If $M_1[\mu] = E[X] = 0$, then $M_2[\mu] = E[X^2] = \mathrm{Var}[X]$. It is also worthwhile noting that if μ_1 and μ_2 are two distributions then, for any $k \geq 0$, $M_k[\mu_1 + \mu_2] = M_k[\mu_1] + M_k[\mu_2]$, i.e., M_k is a linear operator.

An inequality that proves very useful in the sequel is the following.

Lemma 4.4. *For any discrete distribution μ we have $S[\mu]^2 \leq \frac{1}{3} M_2[\mu] M_0[\mu]^3$.*

The proof of Lemma 4.4 is given in Section 5.4.

4.2. Mass redistribution moves.

Definition 4.5 (Moves). A *move* $v = ([a, b], \delta)$ consists of a unit interval $[a, b]$, so $b = a + 1$, and a signed distribution δ on $[a, b]$ with $M_0[\delta] = M_1[\delta] = 0$. A move v can be *applied to* a distribution μ if the signed distribution $\mu' = \mu + \delta$ is a distribution, in which case we denote the result μ' of this application by $v\mu$. We refer to $\frac{a+b}{2}$ as the *center* of the move.

Note that if v is a move and $\mu' = v\mu$, then $M_0[\mu'] = M_0[\mu]$ and $M_1[\mu'] = M_1[\mu]$, and consequently $C[\mu'] = C[\mu]$.

A sequence $V = \langle v_1, v_2, \ldots, v_\ell \rangle$ of moves and an initial distribution μ_0 naturally define a sequence of distributions $\mu_0, \mu_1, \ldots, \mu_\ell$, where $\mu_i = v_i \mu_{i-1}$ for $1 \leq i \leq \ell$. (It is assumed here that v_i can indeed be applied to μ_{i-1}.) We let $V\mu_0 = \mu_\ell$.

Moves and sequences of moves simulate the behavior of *weightless* blocks and stacks. However, the blocks that we are interested in have unit weight. Instead of explicitly taking into account the weight of the blocks, as we briefly do in Section 4.3, it turns out that it is enough for our purposes to impose a natural restriction on the move sequences considered. We start with the following definition.

Definition 4.6 ($\mu_{\max}$). If $\mu_0, \mu_1, \ldots, \mu_\ell$ is a sequence of distributions, and $a \in \mathbb{R}$, we define

$$\mu_{\max}\{x \geq a\} = \max_{0 \leq i \leq \ell} \mu_i\{x \geq a\}.$$

Expressions like $\mu_{\max}\{x > a\}$, $\mu_{\max}\{x \leq a\}$, and $\mu_{\max}\{x < a\}$ are defined similarly.

Definition 4.7 (Weight-constrained sequences). A sequence $V = \langle v_1, v_2, \ldots, v_\ell \rangle$ of moves that generates a sequence $\mu_0, \mu_1, \ldots, \mu_\ell$ of distributions is said to be *weight-constrained* (*with respect to* μ_0) if, for every $a \in \mathbb{R}$, the number of moves in V centered in $[a, \infty)$ is at most $\mu_{\max}\{x \geq a\}$.

In Section 4.3 we will show the following relation between stacks and weight-constrained sequences:

Lemma 4.8. *If there is a stack composed of n blocks of length* 1 *and weight* 1 *that achieves an overhang of d, there is a weight-constrained sequence of moves transforming an initial distribution* μ *with* $M_0[\mu] = \mu\{x \leq 0\} = n$ *and* $\mu\{x > 0\} = 0$ *into a distribution* ν *with* $\nu\{x \geq d - \frac{1}{2}\} \geq 1$.

The main technical result of this paper is the following result for weight-constrained sequences:

Proposition 4.9. *If a distribution* ν *is obtained from a distribution* μ *with* $\mu\{x \leq 0\} = n \geq 1$ *and* $\mu\{x > 0\} = 0$ *by a weight-constrained move sequence, then* $\nu\{x > 6n^{1/3} - \frac{1}{2}\} < 1$.

Theorem 2.3, the main result of this paper, follows immediately from Proposition 4.9 and Lemma 4.8.

For general move sequences we have the following almost tight result, which might be of some independent interest. In particular, it shows that the weight constraint only has a logarithmic effect on the maximal overhang.

Proposition 4.10. *If a distribution* ν *is obtained from a distribution* μ *with* $\mu\{x \leq 0\} = n \geq 1$ *and* $\mu\{x > 0\} = 0$ *by a move sequence of length at most n, then* $\nu\{x > 2n^{1/3}\log_2 n\} < 1$.

4.3. From overhang to mass movement. In this subsection, we will prove Lemma 4.8, capturing the essential relation between stacks and weight-constrained sequences. The moves of Definition 4.5 mimic the effect that a block can have on the collections of forces within a stack. They fail to take into account, however, the fact that the *weight* of a block is "used up" by the move and is then lost. To faithfully simulate the forces between blocks, we need to introduce the slightly modified definition of *lossy moves*. These lossy moves are only introduced to prove Lemma 4.8, and will only be used here in Section 4.3.

Definition 4.11 (Lossy moves). If $v = ([a, b], \delta)$ is a move, then the *lossy* move $v^{\downarrow}$ associated with it is $v^{\downarrow} = ([a, b], \delta^{\downarrow})$, where $\delta^{\downarrow} = \delta - \{(\frac{a+b}{2}, 1)\}$. A lossy move $v^{\downarrow}$ can be applied to a distribution μ if $\mu' = \mu + \delta^{\downarrow}$ is a distribution, in which case we denote the result μ' of this application by $v^{\downarrow}\mu$.

Note that if $v^{\downarrow} = ([a, b], \delta^{\downarrow})$ is a lossy move and $\mu' = v^{\downarrow}\mu$, then $M_0[\mu'] = M_0[\mu] - 1$ and $M_1[\mu'] = M_1[\mu] - \frac{a+b}{2}$. Hence, lossy moves do not preserve total mass or center of mass.

If $V = \langle v_1, v_2, \ldots, v_\ell\rangle$ is a sequence of moves, we let $V^{\downarrow} = \langle v_1^{\downarrow}, v_2^{\downarrow}, \ldots, v_\ell^{\downarrow}\rangle$ be the corresponding sequence of lossy moves. If μ_0 is an initial distribution, we can naturally define the sequence of distributions $\mu_0, \mu_1, \ldots, \mu_\ell$, where $\mu_i = v_i^{\downarrow}\mu_{i-1}$ for $1 \leq i \leq \ell$, obtained by applying $V^{\downarrow}$ to μ_0.

A collection of forces F_i may also be viewed as a mass distribution. The following lemma is now a simple formulation of the definitions and the discussion of Section 2.2:

Lemma 4.12. *Let* $\{B_1, B_2, \ldots, B_n\}$ *be a balanced stack on the table* B_0. *Let* F_i *be a collection of balancing forces acting between* $\{B_0, \ldots, B_i\}$ *and* $\{B_{i+1}, \ldots, B_n\}$, *for*

$0 \le i < n$. *Let* x_{i+1} *be the* x*-coordinate of the left edge of* B_{i+1}. *Then* F_{i+1} *can be obtained from* F_i *by a lossy move in the interval* $[x_{i+1}, x_{i+1} + 1]$.

We can now link stacks to sequences of lossy moves.

Lemma 4.13. *If there is a stack composed of n blocks of length* 1 *and weight* 1 *that achieves an overhang of d, then there is sequence of at most n lossy moves which can be applied to some distribution* μ *with* $M_0[\mu] = \mu\{x \le 0\} = n$ *and* $\mu\{x > 0\} = 0$, *and which finishes with a lossy move centered at* $d - \frac{1}{2}$.

Proof. Let $\{B_1, B_2, \dots, B_n\}$ be a balanced stack and let B_k be a block in it that achieves an overhang of d. As in Lemma 4.12, we let F_i be a collection of balancing forces acting between $\{B_0, \dots, B_i\}$ and $\{B_{i+1}, \dots, B_n\}$. We let $\mu = F_0$ and $\nu = F_k$. It follows from Lemma 4.12 that ν may be obtained from μ by a sequence of k lossy moves. As all the forces in $\mu = F_0$ are forces applied by the table B_0, and as the table supports the weight of the n blocks of the stack, we have $M_0[\mu_0] = \mu\{x \le 0\} = n$ and $\mu\{x > 0\} = 0$. Block B_k corresponds to a lossy move centered at $d - \frac{1}{2}$. ■

The next simple lemma shows that sequences of lossy moves can be easily converted into weight-constrained sequences of moves and distributions that "dominate" the original sequence.

Lemma 4.14. *If* $\mu_0, \mu_1, \dots, \mu_\ell$ *is a sequence of distributions obtained by a sequence of* lossy *moves, then there exists a sequence of distributions* $\mu'_0, \mu'_1, \dots, \mu'_\ell$ *obtained by a* weight-constrained *sequence of moves such that* $\mu'_0 = \mu_0$, *and* $\mu'_i(x) \ge \mu_i(x)$, *for* $1 \le i \le \ell$ *and* $x \in \mathbb{R}$. *Furthermore, if* $v^{\downarrow}_\ell = ([d-1, d], \delta)$ *is the final lossy move in the sequence then* $\mu'_\ell\{x \ge d - \frac{1}{2}\} \ge 1$.

Proof. The sequence $\mu'_0, \mu'_1, \dots, \mu'_\ell$ is obtained by applying the sequence of moves with which the original sequence of lossy moves is associated. More formally, if $\mu_i = v^{\downarrow}_i \mu_{i-1}$, we let $\mu'_i = v_i \mu'_{i-1}$. If $v_i = ([a - \frac{1}{2}, a + \frac{1}{2}], \delta)$, then μ'_i now has an extra mass of size 1 at a. This mass is *frozen*, and will not be touched by subsequent moves. Hence, if k moves have their centers at or beyond position a, then $\mu'_{\max}\{x \ge a\} \ge \mu'_\ell\{x \ge a\} \ge k$, as required by the definition of weight-constrained sequences. We note that the move v_ℓ leaves a mass of size 1 at $d - \frac{1}{2}$. ■

Lemma 4.8 follows immediately from Lemmas 4.13 and 4.14.

5. BOUNDS ON MASS MOVEMENT PROBLEMS. This section is devoted to the proofs of Propositions 4.9 and 4.10. As mentioned, Proposition 4.9 implies Theorem 2.3, which states that an n-block stack can have an overhang of at most $6n^{1/3}$.

5.1. Moves for a large second moment. In this subsection, we want to show that it takes a lot of moves to convert an initial distribution $\mu = \{(0, 1)\}$ into a distribution ν with mass p at distance at least d from 0, that is, $\nu\{|x| \ge d\} \ge p$. This will later be used to prove that it takes many moves to move a fraction p of the mass out at distance d from the side of the table.

Note that $M_2[\mu] = 0$ while $M_2[\nu] \ge pd^2$. Our analysis is based on this increase in second moment. The precise result of this subsection is as follows.

Lemma 5.1. *If a sequence of ℓ moves transforms $\mu = \{(0, 1)\}$ into a distribution ν then $\ell \geq (3M_2[\nu])^{3/2}$. In particular, if $\nu\{|x| \geq d\} \geq p$, where $d > 0$ and $0 < p < 1$, then $\ell \geq (3p)^{3/2}d^3$.*

In order to prove Lemma 5.1, we introduce special operations called *clearances*, which are only used here in Section 5.1 and for the proof of Lemma 5.3 in Section 5.4.

Definitions 5.2 (Clearances). Associated with each unit interval $[a, b]$ is an operation *clearance* that can be applied to any distribution μ and produces the distribution μ', where $\mu'\{a < x < b\} = 0$, $\mu'(x) = \mu(x)$ for every $x \notin [a, b]$, $M_0[\mu] = M_0[\mu']$, and $M_1[\mu] = M_1[\mu']$. In other words, the clearance operation moves all the mass in the interval $[a, b]$ into the endpoints of this interval while maintaining the center of mass. If v is a move on an interval $[a, b]$, we let $\bar{v}$ denote the clearance associated with $[a, b]$. If V is a sequence of moves, we let $\bar{V}$ denote the corresponding sequence of clearances.

It is a simple observation that, for any move v and any distribution μ, there is a unique move v' such that $v'\mu = \bar{v}\mu$.

Closely related to Lemma 4.4 is the following lemma.

Lemma 5.3. *If μ_1 is obtained from μ_0 by a clearance then*

$$S[\mu_1] - S[\mu_0] \geq 3(M_2[\mu_1] - M_2[\mu_0])^2.$$

The proof of Lemma 5.3 is deferred to Section 5.4.

In the rest of this subsection, we will prove the following two lemmas.

Lemma 5.4. *If a sequence of ℓ moves transforms a distribution μ into a distribution ν, then there is a corresponding sequence of ℓ clearances transforming μ into a distribution $\bar{\nu}$ such that $M_2[\nu] \leq M_2[\bar{\nu}]$.*

Lemma 5.5. *If a sequence of ℓ clearances transforms $\mu = \{(0, 1)\}$ into a distribution $\bar{\nu}$ then $\ell \geq (3M_2[\bar{\nu}])^{3/2}$.*

Combining the two lemmas, we immediately get Lemma 5.1.

5.1.1. Clearances and splits. We will now prove Lemma 5.4 by showing that the obvious sequence of clearances gives a distribution $\bar{\nu}$ satisfying the inequality. The correspondence is simple to describe, but it is less simple to prove that it can only increase the second moment.

We are going to prove the following specific form of Lemma 5.4:

Lemma 5.6. *If V is a sequence of moves that can be applied to μ, then $M_2[V\mu] \leq M_2[\bar{V}\mu]$.*

In order to prove Lemma 5.6, we define *splitting*, which induces a natural partial order on distributions. This is only used here in Section 5.1.1.

Definition 5.7 (Splitting). Let μ and μ' be two distributions. We say that *μ' is a basic split of μ*, denoted $\mu \preceq_1 \mu'$, if μ' is obtained by taking one of the point masses (x_i, m_i) of μ and replacing it by a collection $\{(x'_1, m'_1), \ldots, (x'_\ell, m'_\ell)\}$ of point masses with total mass m_i and center of mass at x_i. We say that μ *splits into* μ', denoted $\mu \preceq \mu'$, if μ' can be obtained from μ by a sequence of zero or more basic splits.

To prove Lemma 5.6 we will show (1) that $\mu \preceq \mu'$ implies $M_2[\mu] \leq M_2[\mu']$ and (2) that $V\mu \preceq \bar{V}\mu$.

The following two lemmas summarize simple properties of splits and clearances that will be explicitly or implicitly used in this section. Their obvious proofs are omitted.

Lemma 5.8.

(i) *If $\mu \preceq \mu'$ and $\mu' \preceq \mu''$, then $\mu \preceq \mu''$.*

(ii) *If $\mu_1 \preceq \mu_1'$ and $\mu_2 \preceq \mu_2'$, then $\mu_1 + \mu_2 \preceq \mu_1' + \mu_2'$.*

(iii) *For any distribution μ we have $\{(C[\mu], M_0[\mu])\} \preceq \mu$.*

(iv) *If $\mu = \{(x_1, m_1), (x_2, m_2)\}$ and $\mu' = \{(x_1', m_1'), (x_2', m_2')\}$, where $x_1' \leq x_1 \leq x_2 \leq x_2'$, $M_0[\mu] = M_0[\mu']$ and $C[\mu] = C[\mu']$, then $\mu \preceq \mu'$.*

Lemma 5.9.

(i) *If $v\mu$ is defined then $v\mu \preceq \bar{v}\mu$.*

(ii) *If $\bar{v}$ is a clearance then $\mu \preceq \bar{v}\mu$.*

(iii) *If $\bar{v}$ is a clearance then $\bar{v}(\mu_1 + \mu_2) = \bar{v}\mu_1 + \bar{v}\mu_2$.*

The following lemma shows that splitting increases the second moment.

Lemma 5.10. *If $\mu \preceq \mu'$ then $M_2[\mu] \leq M_2[\mu']$.*

Proof. Due to the linearity of M_2 and the fact that $\preceq$ is the transitive closure of $\preceq_1$, it is enough to prove the claim when $\mu = \{(x, m)\}$ is composed of a single mass and $\mu' = \{(x_1', m_1'), \ldots, (x_k', m_k')\}$ is obtained from μ by a basic split. For any distribution $\nu = \{(x_1, m_1), \ldots, (x_k, m_k)\}$ and any $c \in \mathbb{R}$, we define $M_2[\nu, c] = \sum_{i=1}^{k} m_i(x_i - c)^2$ to be the second moment of ν about c. As $M_0[\mu] = M_0[\mu']$ and $M_1[\mu] = M_1[\mu']$, a simple calculation shows that $M_2[\mu', c] - M_2[\mu, c] = M_2[\mu'] - M_2[\mu]$, for any $c \in \mathbb{R}$. Choosing $c = x$ and noting that $M_2[\mu, x] = 0$ while $M_2[\mu', x] \geq 0$, we get the required inequality. ■

The next lemma exhibits a relation between clearances and splitting.

Lemma 5.11. *If $\mu \preceq \mu'$ and v is a move that can be applied to μ, then $v\mu \preceq \bar{v}\mu'$.*

Proof. We show that $v\mu \preceq \bar{v}\mu \preceq \bar{v}\mu'$, and use Lemma 5.8(i). The first relation is just Lemma 5.9(i). It remains to show $\bar{v}\mu \preceq \bar{v}\mu'$. By Lemma 5.9(iii), it is enough to prove the claim for $\mu = \{(x, m)\}$ composed of a single mass. Let $[a, b]$ be the interval corresponding to $\bar{v}$. There are two cases. If $x \notin (a, b)$, then

$$\bar{v}\mu = \mu \preceq \mu' \preceq \bar{v}\mu',$$

as required, using Lemma 5.9(ii). The more interesting case is when $x \in (a, b)$. Let $\nu = \bar{v}\mu = \{(a, m_1), (b, m_2)\}$ and $\nu' = \bar{v}\mu'$. Let $\nu_\ell' = \nu'_{(-\infty,a]}$ and $\nu_r' = \nu'_{[b,\infty)}$. As $\bar{v}$ leaves no mass in (a, b), we get that $\nu' = \nu_\ell' + \nu_r'$. Let $\bar{m}_\ell = M_0[\nu_\ell']$, $\bar{m}_r = M_0[\nu_r']$,

$\bar{x}_\ell = C[\nu'_\ell]$, and $\bar{x}_r = C[\nu'_r]$. As $\bar{x}_\ell \le a < b \le \bar{x}_r$, we get, using Lemma 5.8(iv), (iii), and (ii), that

$$\begin{aligned}\nu = \{(a, m_1), (b, m_2)\} &\preceq \{(\bar{x}_\ell, \bar{m}_\ell), (\bar{x}_r, \bar{m}_r)\} \\ &= \{(\bar{x}_\ell, \bar{m}_\ell)\} + \{(\bar{x}_r, \bar{m}_r)\} \preceq \nu'_\ell + \nu'_r = \nu',\end{aligned}$$

as required. ■

Using induction we easily obtain the following:

Lemma 5.12. *If V is a sequence of moves that can be applied to μ, then $V\mu \preceq \bar{V}\mu$.*

Lemma 5.6 follows directly from Lemma 5.12 and Lemma 5.10, and Lemma 5.6 implies Lemma 5.4.

5.1.2. Spread vs. second moment. We will now prove Lemma 5.5 stating that if a sequence of ℓ clearances transforms $\mu = \{(0, 1)\}$ into a distribution ν then $\ell \ge (3M_2[\nu])^{3/2}$. The bound relies heavily on Lemma 4.4, which relates the spread and second moment of a distribution, and on Lemma 5.3, which relates differences in spread to differences in second moments.

Let $\mu = \mu_0, \mu_1, \ldots, \mu_\ell = \nu$ be the sequence of distributions obtained by the sequence of ℓ clearances. Note that $M_0[\mu_i] = 1$ for all i, and that $S[\mu_0] = M_2[\mu_0] = 0$.

Let $h_i = M_2[\mu_i] - M_2[\mu_{i-1}]$, for $1 \le i \le \ell$. Then

$$M_2[\mu_\ell] = M_2[\mu_\ell] - M_2[\mu_0] = \sum_{i=1}^{\ell} h_i.$$

By Lemma 5.3 we get that

$$S[\mu_\ell] = S[\mu_\ell] - S[\mu_0] \ge 3\sum_{i=1}^{\ell} h_i^2.$$

Using the Cauchy-Schwartz inequality to justify the second inequality below, we get

$$S[\mu_\ell] \ge 3\sum_{i=1}^{\ell} h_i^2 \ge 3\frac{(\sum_{i=1}^{\ell} h_i)^2}{\ell} = \frac{3M_2[\mu_\ell]^2}{\ell}.$$

Moreover, by Lemma 4.4 on arbitrary distributions, we have

$$S[\mu_\ell] \le \sqrt{\frac{1}{3}M_2[\mu_\ell]}.$$

Hence

$$\ell \ge \frac{3M_2[\mu_\ell]^2}{S[\mu_\ell]} \ge \frac{3M_2[\mu_\ell]^2}{\sqrt{\frac{1}{3}M_2[\mu_\ell]}} = (3M_2[\mu_\ell])^{3/2}.$$

This completes the proof of Lemma 5.5.

5.2. Mirroring. In Section 5.1, we showed that a lot of moves are needed to move some proportion of the mass outwards from the origin. To relate this to the overhang problem we need to transform this result into a lower bound on the moves needed to move a proportion some distance to the *right* of the origin. To achieve this we use a symmetrization approach showing that for any move sequence which shifts mass to the right there is a sequence of similar length which spreads the initial mass outwards. This new sequence corresponds roughly to interleaving the original sequence with a mirror image of itself. The main technical difficulty comes from making this construction work correctly around the origin.

The main result of this section is:

Lemma 5.13. *Let $\mu_0, \mu_1, \ldots, \mu_\ell$ be a sequence of distributions obtained by applying a sequence of moves to an initial distribution μ_0 with $\mu_0\{x > r\} = 0$. If $\mu_{\max}\{x > r\} \leq m$ and $\mu_{\max}\{x \geq r + d\} \geq pm$, where $d > 1$ and $0 < p < 1$, then the sequence of moves must contain at least $\sqrt{3}p^{3/2}(d - \frac{1}{2})^3$ moves whose centers are in $(r + \frac{1}{2}, \infty)$.*

The lemma follows immediately from the following lemma by shifting coordinates and renormalizing masses.

Lemma 5.14. *Let $\mu_0, \mu_1, \ldots, \mu_\ell$ be a sequence of distributions obtained by applying a sequence of moves to an initial distribution μ_0 with $\mu_0\{x > -\frac{1}{2}\} = 0$. If $\mu_{\max}\left\{x > -\frac{1}{2}\right\} \leq 1$ and $\mu_{\max}\{x \geq d\} \geq p$, where $d > \frac{1}{2}$ and $0 < p < 1$, then the sequence of moves must contain at least $\sqrt{3}p^{3/2}d^3$ moves whose centers are at strictly positive positions.*

Proof. We may assume, without loss of generality, that the first move in the sequence moves some mass from $(-\infty, -\frac{1}{2}]$ into $(-\frac{1}{2}, \infty)$ and that the last move moves some mass from $(-\infty, d)$ to $[d, \infty)$. Hence, the center of the first move must be in $(-1, 0]$ and the center of the last move must be at a positive position.

We shall show how to transform the sequence of distributions $\mu_0, \mu_1, \ldots, \mu_\ell$ into a sequence of distributions $\mu'_0, \mu'_1, \ldots, \mu'_{\ell'}$, obtained by applying a sequence of ℓ' moves, such that $\mu'_0 = \{(0, 1)\}$, $\mu'_{\ell'}\{|x| \geq d\} \geq p$, and such that the number of moves ℓ' in the new sequence is at most three times the number ℓ^+ of positively centered moves in the original sequence. The claim of the lemma would then follow immediately from Lemma 5.1.

The first transformation is "negative truncation", where in each distribution μ_i, we shift mass from the interval $(-\infty, -\frac{1}{2})$ to the point $-\frac{1}{2}$. Formally the resulting distribution $\vec{\mu}_i$ is defined by

$$\vec{\mu}_i(x) = \begin{cases} \mu_i(x) & \text{if } x > -\frac{1}{2} \\ 1 - \mu_i\{x > -\frac{1}{2}\} & \text{if } x = -\frac{1}{2} \\ 0 & \text{if } x < -\frac{1}{2}. \end{cases}$$

Note that the total mass of each distribution is 1 and that $\vec{\mu}_0 = \{(-\frac{1}{2}, 1)\}$. Let $\delta_i = \mu_i - \mu_{i-1}$ be the signed distribution associated with the move that transforms μ_{i-1} into μ_i and let $[c_i - \frac{1}{2}, c_i + \frac{1}{2}]$ be the interval in which it operates. For brevity, we refer to δ_i as the move itself, with its center c_i clear from the context. We now compare the transformed "moves" $\vec{\delta}_i = \vec{\mu}_i - \vec{\mu}_{i-1}$ with the original moves $\delta_i = \mu_i - \mu_{i-1}$. If $c_i > 0$, then δ_i acts above $-\frac{1}{2}$ and $\vec{\delta}_i = \delta_i$. If $c_i \leq -1$, then δ_i acts at or below $-\frac{1}{2}$, so $\vec{\delta}_i$ is null and $\vec{\mu}_i = \vec{\mu}_{i-1}$. In the transformed sequence, we skip all such null moves. The remaining case is when the center c_i of δ_i is in $(-1, 0]$. In this case $\vec{\delta}_i$

acts within $[-\frac{1}{2}, \frac{1}{2}]$, and we view it as centered at 0. However, typically $\vec{\delta}_i$ does not define a valid move as it may change the center of mass. We call such $\vec{\delta}_i$ *semi-moves*. Whenever we have two consecutive semi-moves $\vec{\delta}_i$ and $\vec{\delta}_{i+1}$, we combine them into a single semi-move $\vec{\delta}_i + \vec{\delta}_{i+1}$, taking $\vec{\mu}_{i-1}$ directly to $\vec{\mu}_{i+1}$. The resulting sequence we may call the *cleaned negative truncation*. For simplicity, we reindex the surviving distributions consecutively so that $\vec{\mu}_0, \vec{\mu}_1, \dots, \vec{\mu}_{\ell'}, \ell' \leq \ell$, is the cleaned negative truncation. It contains all the original positively centered moves, and now at least every alternate move is of this type. Since the last move in the original sequence was positively centered we conclude:

Claim 5.15. *The cleaned negative truncation is composed of original positively centered moves and semi-moves (acting within* $[-\frac{1}{2}, \frac{1}{2}]$*). The sequence begins with a semi-move and at most half of its elements are semi-moves.*

Next we create a reflected copy of the cleaned negative truncation. First, we define the reflected copy $\overleftarrow{\mu}_i$ of $\vec{\mu}_i$ by

$$\overleftarrow{\mu}_i(x) = \vec{\mu}_i(-x), \quad \text{for every } x \in \mathbb{R},$$

and define the reflected (semi-)moves $\overleftarrow{\delta}_i = \overleftarrow{\mu}_i - \overleftarrow{\mu}_{i-1}$. We can now define the mirrored distributions

$$\overleftrightarrow{\mu}_{2i} = \vec{\mu}_i + \overleftarrow{\mu}_i,$$
$$\overleftrightarrow{\mu}_{2i+1} = \vec{\mu}_{i+1} + \overleftarrow{\mu}_i.$$

Note that $\overleftrightarrow{\mu}_0 = \vec{\mu}_0 + \overleftarrow{\mu}_0 = \{(-\frac{1}{2}, 1), (\frac{1}{2}, 1)\}$. The distribution $\overleftrightarrow{\mu}_{2i+1}$ is obtained from $\overleftrightarrow{\mu}_{2i}$ by the (semi-)move $\vec{\delta}_{i+1}$, and the distribution $\overleftrightarrow{\mu}_{2i+2}$ is obtained from $\overleftrightarrow{\mu}_{2i+1}$ by the (semi-)move $\overleftarrow{\delta}_{i+1}$. Now comes a key observation.

Claim 5.16. *If* $\vec{\delta}_i$ *and* $\overleftarrow{\delta}_i$ *are semi-moves on* $[-\frac{1}{2}, \frac{1}{2}]$, *then their sum* $\overleftrightarrow{\delta}_i = \vec{\delta}_i + \overleftarrow{\delta}_i$ *defines an ordinary move centered at* 0 *and acting on* $[-\frac{1}{2}, \frac{1}{2}]$.

Proof. Both $\vec{\delta}_i$ and $\overleftarrow{\delta}_i$ preserve the total mass. As $\overleftrightarrow{\delta}_i$ is symmetric about 0, it cannot change the center of mass. ■

As suggested by the above observation, if $\vec{\delta}_i$ and $\overleftarrow{\delta}_i$ are semi-moves, we sum them into a single ordinary move $\overleftrightarrow{\delta}_i = \vec{\delta}_i + \overleftarrow{\delta}_i$ centered at 0, taking us directly from $\overleftrightarrow{\mu}_{2i}$ to $\overleftrightarrow{\mu}_{2i+2}$.

Claim 5.17. *After the above summing of semi-move pairs, we have a sequence of at most* $3\ell^+$ *ordinary moves taking us from* $\overleftrightarrow{\mu}_0$ *to* $\overleftrightarrow{\mu}_{2\ell}$. *Here* ℓ^+ *is the number of positively centered moves in the original sequence. The first move in the sequence is the added* $\overleftrightarrow{\delta}_0$, *acting on* $[-\frac{1}{2}, \frac{1}{2}]$, *centered in* 0, *and taking us to* $\overleftrightarrow{\mu}_2$.

Proof. By Claim 5.15, before the mirroring we had ℓ^+ ordinary moves $\vec{\delta}_i$ and at most ℓ^+ semi-moves acting on $[-\frac{1}{2}, \frac{1}{2}]$. For each ordinary move, we get a reflected move $\overleftarrow{\delta}_i$. For each semi-move, we get the single merged move $\overleftrightarrow{\delta}_i$. By Claim 5.15, $\vec{\delta}_0$ is a semi-move that gets added with its reflection. ■

We now replace the initial distribution $\overleftrightarrow{\mu}_0 = \{(-\frac{1}{2}, 1), (\frac{1}{2}, 1)\}$ by the distribution $\mu'_0 = \{(0, 2)\}$, which has the same center of mass, and replace the first move by $\delta'_1 = \overleftrightarrow{\delta}_1 + \{(-\frac{1}{2}, 1), (0, -2), (\frac{1}{2}, 1)\}$. The distribution after the first move is then again $\overleftrightarrow{\mu}_2$.

We have thus obtained a sequence of at most $3\ell^+$ moves that transforms $\mu'_0 = \{(0, 2)\}$ into a distribution $\nu' = \overleftrightarrow{\mu}_\ell$ with $\nu'\{|x| \geq d\} \geq 2p$. Scaling these distributions and moves by a factor of 2, we get, by Lemma 5.1, that $3\ell^+ \geq (3p)^{3/2}d^3$, as claimed. ■

5.3. Proofs of Propositions 4.9 and 4.10. We prove the following lemma which easily implies Proposition 4.9 from Section 4.2.

Lemma 5.18. *Let $\mu_0, \mu_1, \ldots, \mu_\ell$ be a sequence of distributions obtained by applying a weight-constrained sequence of moves on an initial distribution μ_0 with $\mu_0\{x > r\} = 0$. If for some real $m \geq 1/5$ we have $\mu_{\max}\{x > r\} \leq m$, then $\mu_{\max}\{x > r + 6m^{1/3} - \frac{1}{2}\} = 0$.*

Proof. We start by showing that the claim of the lemma holds for all $m \in [1/5, 1)$. We then show that, for $m \geq 1$, if the claim holds for $m/5$ then it also holds for m. By induction, we get that the claim holds for all $m \geq 1/5$. (More formally, we are showing by induction on k that the claim of the lemma holds for all $m \in [1/5, 5^k)$.)

If $m \in [1/5, 1)$, then as $\mu_{\max}\{x > r\} \leq m < 1$ and the sequence is weight-constrained, there is no move whose center is greater than r. Hence $\mu_{\max}\left\{x > r + \frac{1}{2}\right\} = 0$. Since $6m^{1/3} - \frac{1}{2} \geq 6(1/5)^{1/3} - \frac{1}{2} > \frac{1}{2}$, we get the claim of the lemma. This establishes the base case of the induction.

Suppose, therefore, that $\mu_{\max}\{x > r\} \leq m$, where $m \geq 1$, and that the claim of the lemma holds for $m/5$. (We refer to this last condition as the induction hypothesis.) Let u be the least number which is at least 2 such that $\mu_{\max}\{x > r + u\} \leq m/5$. By the induction hypothesis, with r replaced by $r + u$, we get that

$$\mu_{\max}\left\{x > r + u + 6\left(\frac{m}{5}\right)^{1/3} - \frac{1}{2}\right\} = 0.$$

If $u > 2$, we have $\mu_{\max}\{x \geq r + u\} \geq m/5$. We also have $\mu_0\{x > r\} = 0$ and $\mu_{\max}\{x > r\} \leq m$. By Lemma 5.13, with $d = u > 1$, we get that the sequence must contain at least $\sqrt{3}(1/5)^{3/2}(u - \frac{1}{2})^3 > \frac{1}{7}(u - \frac{1}{2})^3$ moves whose centers are in $(r + \frac{1}{2}, \infty)$. As the sequence of moves is weight-constrained and as $\mu_{\max}\{x > r\} \leq m$, there can be at most m such moves with centers greater than r, i.e.,

$$\frac{1}{7}\left(u - \frac{1}{2}\right)^3 \leq m.$$

Hence

$$u \leq (7m)^{1/3} + \frac{1}{2}.$$

Since $m \geq 1$, this bound is greater than 2, so it also holds when $u = 2$. Thus

$$u + 6\left(\frac{m}{5}\right)^{1/3} - \frac{1}{2} \leq \left(7^{1/3} + 6 \cdot \left(\frac{1}{5}\right)^{1/3}\right) m^{1/3} < 5.5m^{1/3} \leq 6m^{1/3} - \frac{1}{2}.$$

This proves the induction step and completes the proof. ■

Modulo the proofs of Lemmas 4.4 and 5.3, which are given in the next section, this completes the proof of our main result that the maximum overhang that can be achieved using n blocks is at most $6n^{1/3}$. It is fairly straightforward to modify the proof of Lemma 5.18 above so as to obtain the stronger conclusion that $\mu_{\max}\left\{x \geq cn^{1/3} - \frac{1}{2}\right\} < 1$, for any $c > 5^{5/2}/(2 \cdot 3^{5/3}) \simeq 4.479$, at least for large enough values of n, and hence an improved upper bound on overhang of $4.5n^{1/3}$, say. In the modified proof we would choose u to be the least number for which $\mu_{\max}\{x \geq r + u\} \geq (27/125)m$. (The constant 27/125 here is the optimal choice.) The proof, however, becomes slightly messier, as several of the inequalities do not hold for small values of m.

Next, we prove the following lemma which easily implies Proposition 4.10.

Lemma 5.19. *Let $\mu_0, \mu_1, \ldots, \mu_n$ be a sequence of distributions obtained by applying a sequence of n moves to an initial distribution μ_0 with $\mu_0\{x > 0\} = 0$ and $\mu_0\{x \leq 0\} = n$. Then $\mu_n\{x > 2n^{1/3}\log_2 n\} < 1$.*

Proof. The result is immediate when $n = 1$ and it is easy to verify whenever $n < 2n^{1/3}\log_2 n$, i.e., for $2 \leq n \leq 31$. From now on we assume just that $n \geq 8$.

Let $k = \lfloor \log_2 n \rfloor + 1 > \log_2 n$. Define $u_0 = 0$ and, for $i = 1, \ldots, k$, let u_i be the least number which is at least $u_{i-1} + 2$ and such that $\mu_{\max}\{x > u_i\} \leq n/2^i$. If $u_i > u_{i-1} + 2$, we have $\mu_{\max}\{x \geq u_i\} \geq n/2^i \geq \mu_{\max}\{x > u_{i-1}\}/2$. By Lemma 5.13, applied with $r = u_{i-1}$, $d = u_i - u_{i-1} > 1$, and $p = \frac{1}{2}$, and noting that the total number of moves is n, we find that

$$\sqrt{3}p^{3/2}\left(d - \frac{1}{2}\right)^3 \leq n,$$

i.e.,

$$u_i - u_{i-1} \leq \frac{\sqrt{2}}{3^{1/6}}n^{1/3} + \frac{1}{2} < \frac{3}{2}n^{1/3}$$

since $n \geq 4$. This bound on $u_i - u_{i-1}$ also holds if $u_i - u_{i-1} = 2$, so we conclude that $u_i - u_{i-1} < \frac{3}{2}n^{1/3}$ for all i. Then

$$u_k < \frac{3kn^{1/3}}{2} = \frac{3n^{1/3}}{2}(\lfloor \log_2 n \rfloor + 1) \leq 2n^{1/3}\log_2 n,$$

since $n \geq 8$. Because $\mu_{\max}\{x > u_k\} \leq n/2^k < 1$, this completes the proof. ∎

As before, the constants in the above proof are not optimized. We believe that a stronger version of the lemma, which states under the same conditions that $\mu_n\{x > cn^{1/3}(\log_2 n)^{2/3}\} < 1$, for some $c > 0$, actually holds. This would match an example supplied by Johan Håstad. Lemma 5.19 (and Proposition 4.10) imply an almost tight bound on an interesting variant of the overhang problem that involves weightless blocks, as discussed in Section 6.

5.4. Proof of spread vs. second moment inequalities.

Lemma 4.4. (The proof was deferred from Section 4.1.) *For any discrete distribution μ,*

$$S[\mu]^2 \leq \frac{1}{3}M_2[\mu]M_0[\mu]^3.$$

The method of proof used here was suggested to us by Benjy Weiss, and resulted in a much improved and simplified presentation. The lemma is essentially the case $n = 2$ of a more general result proved by Plackett [**16**].

Proof. Suppose that $\mu = \{(x_1, m_1), ..., (x_k, m_k)\}$ where $x_1 < x_2 < \cdots < x_k$.

We first transform the coordinates into a form which will be more convenient for applying the Cauchy-Schwartz inequality. Since the statement of the lemma is invariant under scaling of the masses, we may assume that $M_0[\mu] = 1$. Also, since S is invariant under translation and M_2 is minimized by a translation which moves $C[\mu]$ to the origin, we may assume without loss of generality that $C[\mu] = 0$.

Define a function $g(t)$ for $-\frac{1}{2} < t \leq \frac{1}{2}$ by

$$g(t) = x_i, \quad \text{where} \quad \sum_{r=1}^{i-1} m_r < t + \frac{1}{2} \leq \sum_{r=1}^{i} m_r,$$

and define $g(-\frac{1}{2}) = x_1$. Thus the value of g is x_1 on $[-\frac{1}{2}, -\frac{1}{2} + m_1]$, x_2 on $(-\frac{1}{2} + m_1, -\frac{1}{2} + m_1 + m_2]$, etc.

Now we have that

$$M_j[\mu] = \sum_{i=1}^{k} x_i^j m_i = \int_{t=-1/2}^{1/2} g(t)^j \, dt$$

for $j \geq 0$, and

$$S[\mu] = \sum_{i<j} m_i m_j (x_j - x_i) = \int_{t=-1/2}^{1/2} \int_{s=-1/2}^{t} (g(t) - g(s)) \, ds \, dt.$$

Above it may seem that the integral should have been restricted to the case where $g(s) < g(t)$. However, if $g(t) = g(s)$, the integrand is zero, so this case does not contribute to the value of the integral.

Recall that $C[\mu] = 0$, so that $M_1[\mu] = \int_{t=-1/2}^{1/2} g(t) \, dt = 0$. Therefore

$$\int_{t=-1/2}^{1/2} \int_{s=-1/2}^{t} g(t) \, ds \, dt = \int_{t=-1/2}^{1/2} \left(t + \frac{1}{2}\right) g(t) \, dt = \int_{t=-1/2}^{1/2} t g(t) \, dt,$$

while

$$\int_{t=-1/2}^{1/2} \int_{s=-1/2}^{t} g(s) \, ds \, dt = \int_{s=-1/2}^{1/2} \int_{t=s}^{1/2} g(s) \, dt \, ds$$
$$= \int_{s=-1/2}^{1/2} \left(\frac{1}{2} - s\right) g(s) \, ds = -\int_{s=-1/2}^{1/2} s g(s) \, ds.$$

So

$$S[\mu] = 2 \int_{t=-1/2}^{1/2} t g(t) \, dt. \tag{$\dagger$}$$

Using the Cauchy-Schwartz inequality,

$$S[\mu]^2 = 4\left(\int_{t=-1/2}^{1/2} tg(t)\,dt\right)^2 \le 4\int_{t=-1/2}^{1/2} g(t)^2\,dt \cdot \int_{t=-1/2}^{1/2} t^2\,dt$$
$$= 4M_2[\mu]\cdot\frac{1}{12} = \frac{1}{3}M_2[\mu]. \qquad \blacksquare$$

Lemma 5.3. (The proof was deferred from Section 4.2.) *If μ_1 is obtained from μ_0 by a clearance then*

$$S[\mu_1] - S[\mu_0] \ge 3(M_2[\mu_1] - M_2[\mu_0])^2.$$

Proof. For any move, the resulting changes in spread and second moment are invariant under linear translation of the coordinates, so we may assume that the interval of the move is $[-\frac{1}{2}, \frac{1}{2}]$. These differences are also independent of any weight outside the interval of the move, so we may assume that μ_0 has all its support in $[-\frac{1}{2}, \frac{1}{2}]$.

Since our move is a clearance, the addition of an extra point mass at either $-\frac{1}{2}$ or $\frac{1}{2}$ leaves the differences in spread and second moment invariant. We may therefore add such a mass to bring the center of mass of μ_0 to 0. Finally, since the statement of the lemma is invariant under scaling of the masses, we may further assume that $M_0[\mu_0] = 1$.

Since the clearance pushes all the mass to the endpoints of the interval, we get

$$\mu_1 = \left\{\left(-\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right)\right\} \quad \text{and} \quad M_2[\mu_1] = S[\mu_1] = \frac{1}{4}.$$

We define $g(t)$ for $-\frac{1}{2} \le t \le \frac{1}{2}$ just as in the proof of Lemma 4.4, but now corresponding to the distribution μ_0. As before, $M_j[\mu_0] = \int_{-1/2}^{1/2} g(t)^j\,dt$ for $j \ge 0$, and we recall as in (†) that $S[\mu_0] = 2\int_{-1/2}^{1/2} tg(t)\,dt$.

We have $M_1[\mu_0] = \int_{-1/2}^{1/2} g(t)\,dt = 0$. Let $c = M_2[\mu_0] = \int_{-1/2}^{1/2} g(t)^2\,dt$ and $s = S[\mu_0]$. By Lemma 4.4 we have $s^2 \le \frac{c}{3}$. If $c \le \frac{1}{12}$ then $s \le \sqrt{\frac{c}{3}} \le \frac{1}{6}$, and the result follows immediately as

$$\begin{aligned} S[\mu_1] - S[\mu_0] - 3(M_2[\mu_1] - M_2[\mu_0])^2 &= \frac{1}{4} - s - 3\left(\frac{1}{4} - c\right)^2 \\ &\ge \frac{1}{4} - s - 3\left(\frac{1}{4} - 3s^2\right)^2 \\ &= \frac{(1+2s)(1-6s)^3}{16} \ge 0. \end{aligned}$$

We next claim that if $c = M_2[\mu_0] > \frac{1}{12}$, then $s = S[\mu_0] \le \frac{1}{4} - \frac{a^2}{12}$, where $a = \frac{3}{2} - 6c < 1$. To prove this claim, we define a function $h(t)$ as follows:

$$h(t) = \begin{cases} \frac{t}{a} & \text{if } |t| \le \frac{a}{2}, \\ \frac{1}{2}\operatorname{sgn}(t) & \text{otherwise.} \end{cases}$$

We may verify that

$$\int_{-1/2}^{1/2} h(t)^2\, dt = \frac{1}{4} - \frac{a}{6} = c \quad \text{and} \quad \int_{-1/2}^{1/2} t\, h(t)\, dt = \frac{1}{8} - \frac{a^2}{24}.$$

By the Cauchy-Schwartz inequality,

$$\left(\int_{-1/2}^{1/2} h(t)g(t)\, dt\right)^2 \le \int_{-1/2}^{1/2} h(t)^2\, dt \cdot \int_{-1/2}^{1/2} g(t)^2\, dt = c^2,$$

and so

$$\int_{-1/2}^{1/2} h(t)g(t)\, dt \le c = \int_{-1/2}^{1/2} h(t)^2\, dt. \qquad (*)$$

We also have

$$\int_{-1/2}^{1/2} \left(\frac{t}{a} - h(t)\right) g(t)\, dt \le \int_{-1/2}^{1/2} \left(\frac{t}{a} - h(t)\right) h(t)\, dt, \qquad (**)$$

since $h(t) - g(t) \le 0$ and $\frac{t}{a} - h(t) \le 0$ for $t < -\frac{a}{2}$, $h(t) - g(t) \ge 0$ and $\frac{t}{a} - h(t) \ge 0$ for $t > \frac{a}{2}$, and $\frac{t}{a} - h(t) = 0$ for $|t| \le \frac{a}{2}$. Adding inequalities $(*)$ and $(**)$, and multiplying by $2a$, gives

$$S[\mu_0] = 2\int_{-1/2}^{1/2} t\, g(t)\, dt \le 2\int_{-1/2}^{1/2} t\, h(t)\, dt = \frac{1}{4} - \frac{a^2}{12}.$$

Finally,

$$S[\mu_1] - S[\mu_0] \ge \frac{1}{4} - \left(\frac{1}{4} - \frac{a^2}{12}\right) = \frac{a^2}{12} = 3\left(\frac{1}{4} - c\right)^2 = 3(M_2[\mu_1] - M_2[\mu_0])^2.$$

This completes the proof. ■

We end the section by noting that although the inequalities of Lemmas 4.4 and 5.3 are only claimed for discrete distributions, which is all we need in this paper, our proofs can easily be modified to show that they also hold for general continuous distributions. In fact, for nontrivial discrete distributions the inequalities in the two lemmas are always *strict*. In the continuous case, the inequalities are satisfied with equality by appropriately chosen uniform distributions. In particular, the constant factors $\frac{1}{3}$ and 3 appearing in the two lemmas cannot be improved.

6. CONCLUDING REMARKS AND OPEN PROBLEMS. We have shown that the maximum overhang achieved using n homogeneous, frictionless blocks of unit length is at most $6n^{1/3}$. Thus, the constructions of [**14**] cannot be improved by more than a constant factor, establishing order $n^{1/3}$ as the asymptotic answer to the age-old overhang problem.

The discussions and results presented so far all referred to the standard two-dimensional version of the overhang problem. Our results hold, however, in greater generality. We briefly discuss some natural generalizations and variants of the overhang problem for which our bounds still apply.

In Section 2 we stipulated that all blocks have a given height h. It is easy to see, however, that all our results remain valid even if blocks have different heights, but still have unit length and unit weight. In particular, blocks are allowed to degenerate into *sticks*, i.e., have height 0. Also, even though we required blocks not to overlap, we did not use this condition in any of our proofs.

Loaded stacks, introduced in [**14**], are stacks composed of standard unit length and unit weight blocks, and *point weights* that can have arbitrary weight. (Point weights may be considered to be blocks of zero height and length, but nonzero weight.) Our results, with essentially no change, imply that loaded stacks of total weight n can have an overhang of at most $6n^{1/3}$.

What happens when we are allowed to use blocks of different lengths and weights? Our results can be generalized in a fairly straightforward way to show that if a block of length ℓ has weight proportional to ℓ^3, as would be the case if all blocks were similar three-dimensional cuboids, then the overhang of a stack of total weight n is again of order at most $n^{1/3}$. It is amusing to note that in this case an overhang of order $n^{1/3}$ can be obtained by stacking n unit-length blocks as in the construction of [**14**], or simply by balancing a single block of length $n^{1/3}$ and weight n at the edge of the table!

In case the weights are not strictly proportional to the lengths cubed, we could define c to be the smallest constant such that blocks of weight w have length at most $cw^{1/3}$. If the total weight of the stack is W, then its overhang is bounded by $6cW^{1/3}$.

Proposition 4.10 supplies an almost tight upper bound for the following variant of the overhang problem: how far away from the edge of a table can a mass of weight 1 be supported using n *weightless* blocks of length 1 and a collection of point weights of total weight n? The overhang in this case beats the classical one by a factor of between $\log^{2/3} n$ and $\log n$.

In all variants considered so far, blocks were assumed to have their largest faces parallel to the table's surface and perpendicular to its edge. The assumption of no friction then immediately implied that all forces within a stack are vertical, and our analysis, which assumes that there are no horizontal components, was applicable. A nice argument, communicated to us by Harry Paterson, shows that in the frictionless two-dimensional case, no blocks can lean against each other inducing horizontal forces. We could have a block balanced on its corner, but this would not create nonvertical forces. Our results thus apply also to this general two-dimensional case where the blocks may be stacked arbitrarily.

We believe that our bounds also apply, with slightly adjusted constants, in three dimensions, but proving so remains an open problem. Overhang larger by a factor of

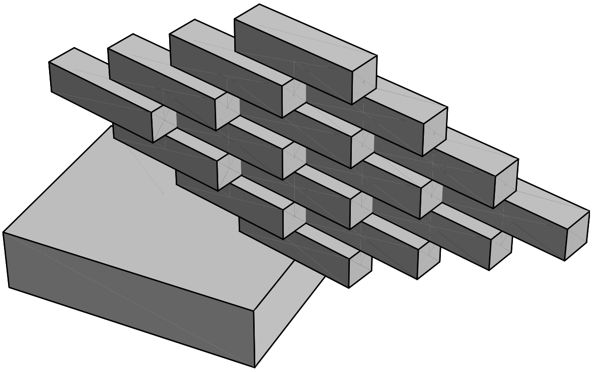

Figure 8. A "skintled" 4-diamond.

$\sqrt{1+w^2}$ may be obtained with $1 \times w \times h$ blocks, where $h \leq w \leq 1$, using a technique called *skintling* (see Figure 8). In skintling (a term we learned from an edifying conversation with John H. Conway about brick-laying), each block is rotated about its vertical axis, so that—in our case—the diagonal of its bottom face is perpendicular to the edge of the table. With length defined by the projection on the direction of the overhang, our bounds apply to any three-dimensional construction that can be balanced using vertical forces only. It is an interesting open problem whether there exist three-dimensional stacks composed of frictionless, possibly tilted, homogeneous rectangular blocks that can only be balanced with the aid of some nonvertical forces. Figure 9 shows that this is possible with some convex homogeneous blocks of different shapes and sizes. As mentioned, we believe that our bounds do apply in three dimensions for regular blocks even if it turns out that nonvertical forces are sometimes useful, but proving this requires some additional arguments.

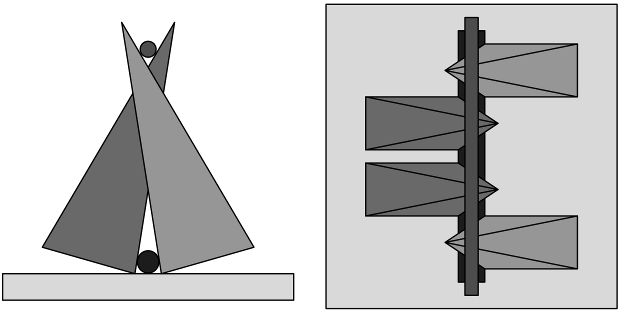

Figure 9. A balanced stack of convex objects with nonvertical forces.

We end by commenting on the tightness of the analysis presented in this paper. Our main result is a $6n^{1/3}$ upper bound on the overhang that may be obtained using n blocks. As mentioned after the proof of Lemma 5.18, this bound can easily be improved to about $4.5n^{1/3}$ for sufficiently large values of n. Various other small improvements in the constants are possible. For example, a careful examination of our proofs reveals that, whenever we apply Lemma 5.3, the distribution μ_0 contains at most three masses in the interval acted upon by the move that produces μ_1. (This follows from the fact that a block can rest upon at most three other blocks.) The constant 3 appearing in Lemma 5.3 can then be improved, though it is optimal when no assumption regarding the distribution μ_0 is made. Finally, we note our bound only counts blocks with centers strictly over the side of the table; that is, if there are at most n such blocks over the side, then the overhang is proved to be at most $6n^{1/3}$. We believe, however, that new ideas would be needed to reduce the upper bound to below $3n^{1/3}$, say.

As mentioned, Paterson and Zwick [**14**] describe simple balanced n-block stacks that achieve an overhang of about $0.57n^{1/3}$. They also present some numerical evidence that suggests that the overhang achievable using n blocks is at least $1.02n^{1/3}$, for large values of n. These larger overhangs are obtained using stacks that are shaped like the "oil lamp" depicted in Figure 10. For more details on the figure and on "oil-lamp" constructions, see [**14**]. (The stack shown in the figure is actually a loaded stack, as defined above, with the external forces shown representing the point weights.)

A small gap still remains between the best upper and lower bounds currently available for the overhang problem, though they are both of order $n^{1/3}$. Determining a constant c such that the maximum overhang achievable using n blocks is asymptotically $cn^{1/3}$ is a challenging open problem.

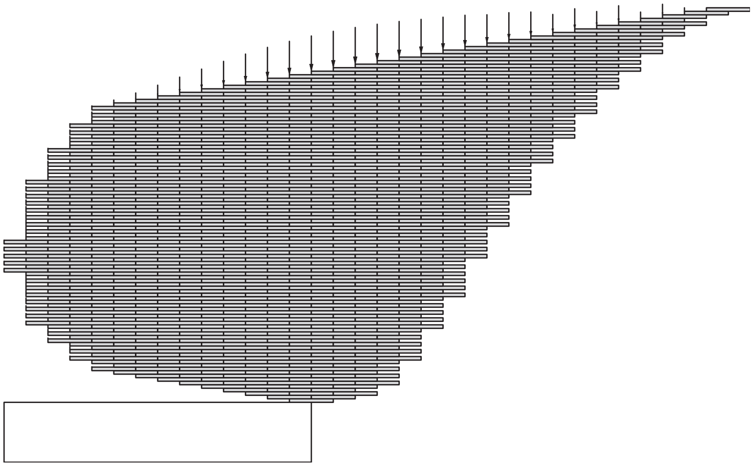

Figure 10. An "oil-lamp" loaded stack with overhang 10 having 921 blocks and total weight 1112.88.

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