Statistics; Finite Mathematics

Combinations and Their Sums

Suppose that you work in an office of 9 people, and 2 of you are to be sent to a professional development seminar. How many possible outcomes are there for choosing the 2 people? Since the 2 people must be different from one another, and since the order in which they're chosen doesn't matter, the number of possible outcomes is the number of **combinations** of 9 people taken 2 at a time:

$n = 9$ employees eligible for the seminar
$r = 2$ employees chosen for the seminar
} without repeats, without order

ans $C(n,r) = C(9,2)$

$$= \frac{9!}{2!7!}$$

$$= \frac{9\times 8\times 7\times 6\times \cdots \times 1}{(2\times 1)(7\times 6\times \cdots \times 1)}$$

$$= \frac{9\times 8}{2} \quad \text{[note shortcut for } C(n,2)\text{]}$$

$= 36$ possible outcomes

Calculations like this one have been carried out by people solving problems across many different centuries and cultures. The problems have ranged from practical ones to others posed just for interest or practice. Use the factorial formula, or the nCr feature of your calculator, to work the following exercises that have been drawn from various cultures.

Exercise 1. In his encyclopedic work *Brhatsamhitā*, written in the city of Ujjain in central India in the 500s, the astronomer/mathematician Varāhamihira listed 16 fragrances that can be used in perfumes. He then calculated the number of different perfumes incorporating 4 of the 16 fragrances. Predict the result of his calculation:

$n = 16$ fragrances available
$r = 4$ fragrances chosen
} without repeats, without order

ans $C(\_\_\_\_, \_\_\_\_) =$ _______________

Rheinisches Landesmuseum Bonn

Exercise 2. Permutations and combinations came to be of special interest to mathematicians of the Jaina religion, including one named Mahāvīra who lived in Mysore, southern India in the 800s. At one point in his nine-chapter *Compendium of the Essence of Mathematics* (*Ganita Sāra Sangraha*), Mahāvīra counted the number of different types of jeweled necklaces that can be made, depending on the choice of diamonds, sapphires, emeralds, corals, and/or pearls. In how many different ways can 3 of these types of jewels be selected?

exs. {diamonds, emeralds, corals}
{sapphires, corals, pearls}

$n =$ ____________________
$r =$ ____________________
}

ans $C(\_\_\_\_, \_\_\_\_) =$ _______________

Necklace photo by J. M. Kenoyer, http://www.harappa.com/indus.

Courtesy Dept. of Archaeology and Museums, Govt. of Pakistan

Schwartz, Randy K., "Combining Strands of Many Colors: Episodes from Medieval Islam for the Mathematics Classroom," *Loci: Convergence* (August 2010), DOI: 10.4169/loci003546

Exercise 3. Abraham ibn Ezra (1092-1167) was a Jewish rabbi living in Spain. Throughout the Middle Ages, much of Spain and Portugal was ruled by Moors (Arab and Berber Muslims from Morocco and lands further south and east). Ibn Ezra helped introduce their Arab numerals, arithmetic and algebra to Europe. For example, in an astrological treatise called *The Book of the World* (*Se'fer Ha'Olam*), ibn Ezra used combinations to calculate the number of possible types of planetary conjunctions. In his day, all of the "planets" (bodies that wander the heavens) were thought to circle the Earth: these were the Moon, Mercury, Venus, the Sun, Mars, Jupiter and Saturn. When two or more of these planets appear to meet in the sky, it is called a "conjunction." How many different types of conjunctions of 4 of these 7 planets are there?

<u>exs.</u> { Sun, Moon, Venus, Jupiter}
{ Mercury, Venus, Mars, Saturn }

n = ___________________
r = ___________________

<u>ans</u> C(____, ____) = _________

Drawing of the geocentric concept of the universe, with Earth at the center surrounded by seven planetary spheres. From Peter Apian, *Cosmographia* (Antwerp, Belgium, 1539)

Image courtesy History of Science Collections, University of Oklahoma Libraries

Exercise 4. Al-Samaw'al ben Yahyā (c. 1125-1180) was a Jewish doctor and mathematician in Baghdad who learned Arabic and later converted to Islam. When he was only 19, he wrote an algebra text called *The Shining Book of Calculation* (*Al-Bāhir fi-l-hisāb*). At one point in the text, he discussed how to solve systems of many linear equations and variables, and he actually listed all possible combinations of 10 variables taken 6 at a time. Predict how many he listed:

n = ___________________
r = ___________________

<u>ans</u> C(____, ____) = _________

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A mathematician and doctor living in Marrakech, Morocco around 1200, **Ahmad al-Ab'darī ibn Mun'im**, posed and answered a series of questions about how to make silk tassels by combining strands of thread of various colors. He asked,

> How many different tassels can be made from 3 colors, if 10 different colors are available?

Nowadays, we might simply write down the answer as $C(10,3)$. But ibn Mun'im didn't yet have a formula for $C(n,3)$ but only for $C(n,2)$. So he thought it through a different way, and discovered something important in the process. Start with a list of the 10 different colors (sample at right). He reasoned,

color 10 gold
color 9 purple
color 8 red
color 7 azure
color 6 white
color 5 orange
color 4 silver
color 3 emerald
color 2 black
color 1 indigo

Photo: Laurance Tovrea

To select 3 of the 10 colors, we can choose *either*:

- ❖ color 3 and both of the 2 colors below it, *or*
- ❖ color 4 and any 2 of the 3 colors below it, *or*
- ❖ color 5 and any 2 of the 4 colors below it, *or*
- ❖ color 6 and any 2 of the 5 colors below it, *or*
- ❖ color 7 and any 2 of the 6 colors below it, *or*
- ❖ color 8 and any 2 of the 7 colors below it, *or*
- ❖ color 9 and any 2 of the 8 colors below it, *or*
- ❖ color 10 and any 2 of the 9 colors below it.

In other words, ibn Mun'im classified each tassel according to how high up it went in this color list.

Exercise 5. Would you agree that these choices are mutually exclusive and mutually exhaustive?______________

But what if someone came up to you and said, "Wait a minute, you've missed all sorts of tassels. For example, what about a tassel that uses color 6 and any 2 of the 4 colors *above* it?" How would you refute her/his objection?

Let's look again at the list of ways to select 3 of the 10 colors. Since these choices are mutually exclusive and mutually exhaustive, we can add them up to get the total number of choices, as shown here.

To select 3 of the 10 colors, we can choose *either*:

- ❖ color 3 and both of the 2 colors below it, *or* [there are $C(2,2)$ ways to do that]
- ❖ color 4 and any 2 of the 3 colors below it, *or* [there are $C(3,2)$ ways to do that]
- ❖ color 5 and any 2 of the 4 colors below it, *or* [there are $C(4,2)$ ways to do that]
- ❖ color 6 and any 2 of the 5 colors below it, *or* [there are $C(5,2)$ ways to do that]
- ❖ color 7 and any 2 of the 6 colors below it, *or* [there are $C(6,2)$ ways to do that]
- ❖ color 8 and any 2 of the 7 colors below it, *or* [there are $C(7,2)$ ways to do that]
- ❖ color 9 and any 2 of the 8 colors below it, *or* [there are $C(8,2)$ ways to do that]
- ❖ color 10 and any 2 of the 9 colors below it. [there are $C(9,2)$ ways to do that]

---

so $C(10,3) = C(2,2) + C(3,2) + C(4,2) + C(5,2) + C(6,2) + C(7,2) + C(8,2) + C(9,2)$.

Exercise 6.

a) Use the shortcut formula for $C(n,2)$ to complete ibn Mun‘im’s calculation:

$C(10,3) =$ _____+ _____+ _____+ _____+ _____+ _____+ _____+ _____

$C(10,3) =$ _____

b) Using the factorial formula, or the nCr feature of your calculator, do you get the same answer?

$C(10,3) =$ _____

Exercise 7. Use the same logic as ibn Mun‘im to write similar expansions.

ex. $C(10,4) = \underline{C(3,3)} + \underline{C(4,3)} + \underline{C(5,3)} + \underline{C(6,3)} + \underline{C(7,3)} + \underline{C(8,3)} + \underline{C(9,3)}$

To choose 4 of the 10 colors, choose either color 4 and all 3 of the 3 colors below it, or color 5 and any 3 of the 4 colors below it, ……, or color 10 and any 3 of the 9 colors below it.

a) $C(10,6) = \underline{C(5,5)} +$ _______+ _______+ _______+ _______

b) $C(10,7) =$ _______+ _______+ _______+ _______

c) $C(8,5) =$ _______+ _______+ _______+ _______

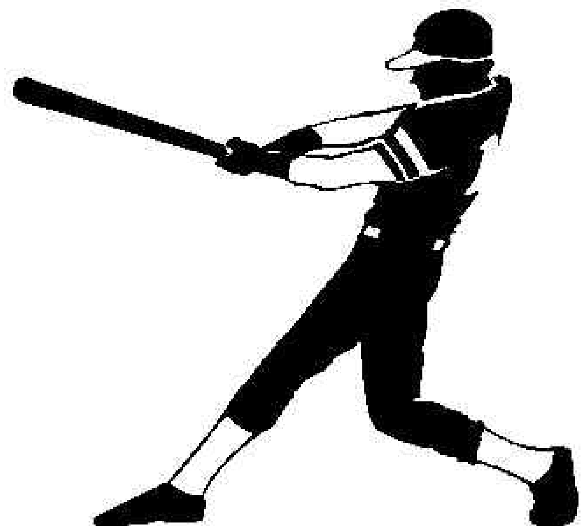
d) $C(7,5) =$

Exercise 8. Interpret (in words) your answer to Exercise 7b in terms of making a tassel from silk threads of various colors. Model your phrasing after the example on page 3.

Exercise 9. Interpret (in words) your answer to Exercise 7c in terms of selecting professional development trainees from among the people in an office.

Exercise 10. Use numbers and words of your own choosing to make up a problem like that in Exercises 8 and 9, and provide an answer for it.

Exercise 11. Let's count the number of ways that the American League ("A") team could defeat the National League ("N") team to win the Major League Baseball World Series. The winner is the first team to win 4 games, so the series lasts between 4 and 7 games.



exs. AAAA ("A" team sweeps the series with 4 straight wins)

ANAANA (6-game series)

NNAAANA (7-game series)

To win the series, the "A" team must win the final game as well as 3 games leading up to that. So, for example, the number of possible 7-game series is $C(6,3)$, the number of different ways to select 3 wins from among the first 6 games. Reasoning in this way, we get:

no. of ways for "A" team to win series = (no. of possible 4-game series) + … + (no. of possible 7-game series)

$$= C(3,3) + C(4,3) + C(5,3) + C(6,3)$$

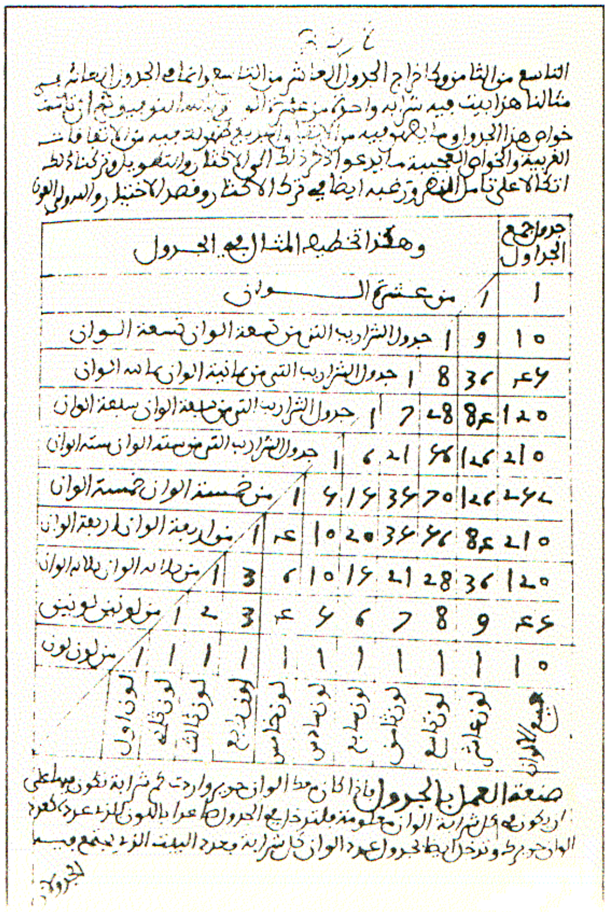
$$= C(\_\_\_\_, \_\_\_\_)$$

$$= \_\_\_\_\_\_$$

Thanks to ibn Mun'im's shortcut, the entire calculation can be done with a single C(n,r) !

In addition to using words, ibn Mun'im's discovery about sums of combinations can also be summarized in two other ways:

- Using algebraic symbols: $C(n,r) = C(r-1,r-1) + C(r,r-1) + C(r+1,r-1) + \cdots + C(n-1,r-1)$.
- Graphically, ibn Mun'im used a triangular chart to tally up all the different tassels that can be formed from 10 colors or less. His chart, or **arithmetical triangle** (shown below on the left), uses the Indo-Arabic numerals as they were written in his day. On the right is a translation into our familiar notation.



For a tassel of 10 colors										1	1
For a tassel of 9 colors									1	9	10
For a tassel of 8 colors								1	8	36	45
For a tassel of 7 colors							1	7	28	84	120
For a tassel of 6 colors						1	6	21	56	126	210
For a tassel of 5 colors					1	5	15	35	70	126	252
For a tassel of 4 colors				1	4	10	20	35	56	84	210
For a tassel of 3 colors			1	3	6	10	15	21	28	36	120
For a tassel of 2 colors		1	2	3	4	5	6	7	8	9	45
For a tassel of 1 color	1	1	1	1	1	1	1	1	1	1	10
	using color #1	using color #2 (& earlier)	using color #3 (& earlier)	using color #4 (& earlier)	using color #5 (& earlier)	using color #6 (& earlier)	using color #7 (& earlier)	using color #8 (& earlier)	using color #9 (& earlier)	using color #10 (& earlier)	Total no. of tassels

Image: *Actes du Huitième Colloque Maghrébin sur l'Histoire des Mathématiques Arabes, Tunis, les 18-19-20 Décembre 2004* (Tunis: Tunisian Association of Mathematical Sciences, 2006)

<u>ex.</u> Using ibn Mun'im's chart, how many different tassels can be formed by selecting 3 of the 10 colors?

We use the row labeled "For a tassel of 3 colors." As we saw on page 3, we can do this using color #3 (and earlier colors), or using color #4 (and earlier colors), etc. In each case the number of ways to do so is given in the chart, arranged along the given row. Let's total them up from left to right:

$$C(10,3) = 1+3+6+10+15+21+28+36$$
$$= 120.$$

This should match your calculation in Exercise 6a above.

Exercise 12. Using ibn Mun'im's chart, how many different tassels can be formed by selecting 5 of the 10 colors?

$C(10,5) =$ _______ + _______ + _______ + _______ + _______ + _______

$=$ _______

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Ibn Mun‘im also pointed out a shortcut that takes advantage of the pattern mentioned earlier. Select any row, starting with the 1 on the left. As you add numbers to its right, notice that at each step, the accumulated sum matches the number in the cell diagonally above and to the right. For example,

				10
For a tassel of 3 colors	1	3	6 ↗	

1 + 3 + 6 = 10

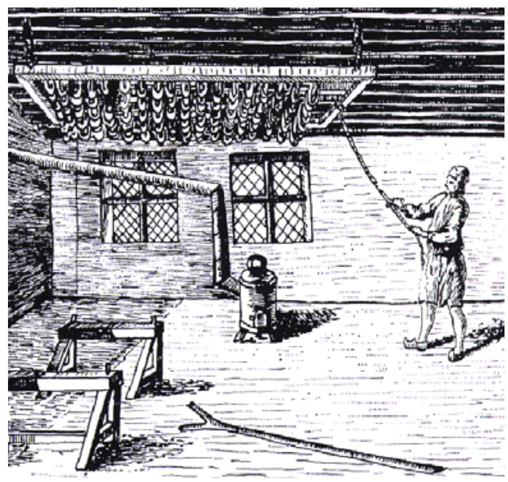
This pattern, which we’ll call the **row-sum pattern**, gives us a fast way to construct or enlarge ibn Mun‘im’s arithmetical triangle. It graphically represents the algebraic pattern mentioned earlier:

$$\underbrace{C(r-1,r-1)+C(r,r-1)+C(r+1,r-1)+\cdots+C(n-1,r-1)}_{\text{These cells lie consecutively on one row.}}=C(n,r)$$

These two cells lie diagonally opposite each other.

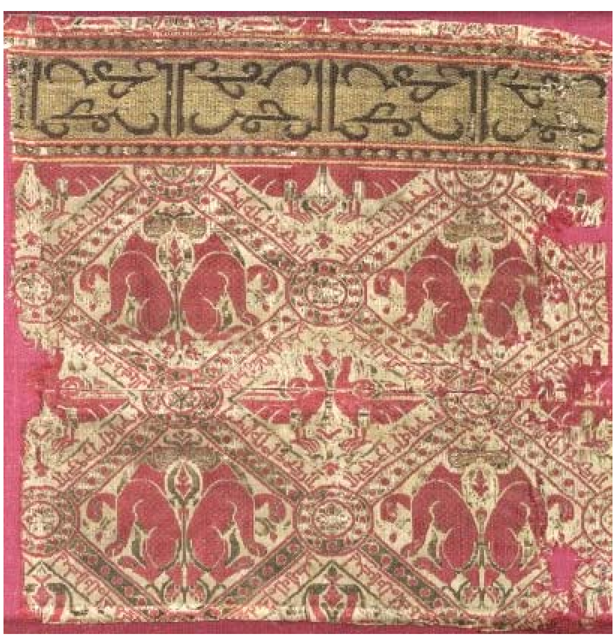
Like all other medieval mathematicians, ibn Mun‘im did not have algebraic symbols; instead, equations were written out *rhetorically*, that is, in words. But here, we are using modern notation to abbreviate his work.

Ibn Mun‘im was born in Dénia, on the east coast of Spain near Valencia. But he spent much of his life in Marrakech at the court of the Almohads, the Moors who ruled much of what is now Morocco, Spain and Portugal. The production of silk and the weaving of silk products in cities like Marrakech and Granada was one of the major industries and sources of wealth of the Almohad empire.



In a production workshop, bundles of silk that have just been dyed with a color pigment are dried on a movable overhead platform.

Image from Manuel Garzon Pareja, *La Industria Sedera en España: El Arte de la Seda de Granada* (Granada: Archivo de la Real Chancillería, 1972).



This fragment of fabric from 12th-Century southern Spain, finely woven from dyed silk, shows the kinds of decoration that were pleasing to the Moorish inhabitants of that region. Peacocks and mythical griffins are paired off within each lozenge shape, and an inscription appears above in Kufic Arabic script. Luxury textiles such as these, very portable compared to heavier or breakable treasures, were often used as diplomatic and royal gifts.

But ibn Mun‘im wasn’t so interested in questions about silk tassels (*sharārīb* in Arabic) for their own sake. These were “warm-up exercises” to prepare for a more difficult feat: to compute the number of words (of various lengths) that can be formed from the 28 letters of the Arabic alphabet. By considering the silk tassel problem, ibn Mun‘im was able to discover and explain important number patterns in the arithmetical triangle. He then used these patterns to solve more difficult problems.

Exercise 13. The verses of a famous song, "The Twelve Days of Christmas," run as follows:

On the 1st day of Christmas my true love gave to me: A Partridge in a Pear Tree.

On the 2nd day of Christmas my true love gave to me: Two Turtle Doves and A Partridge in a Pear Tree.

.
.
.

On the 12th day of Christmas my true love gave to me: Twelve Drummers Drumming, Eleven Pipers Piping, Ten Lords A-leaping, Nine Ladies Dancing, Eight Maids A-milking, Seven Swans A-swimming, Six Geese A-laying, Five Golden Rings, Four Calling Birds, Three French Hens, Two Turtle Doves, and A Partridge in a Pear Tree.

Let's use ibn Mun'im's row-sum pattern to count the total number of gifts received in the Twelve Days. The pattern can be used to find the number of gifts on each day. Then it can be used one last time (shown below the bar) to add up those subtotals to get the grand total.

On Day 1: $1 \quad = C(1,1) \quad = C(2,2)$ by row-sum pattern

On Day 2: $1+2 \quad = C(1,1)+C(2,1) \quad = C(3,2)$ by row-sum pattern

On Day 3: $1+2+3 \quad = C(1,1)+C(2,1)+C(3,1) \quad = C(\_\_\_,\_\_\_)$ by row-sum pattern

.
.
.

On Day 12: $1+2+3+\cdots+12 = C(1,1)+C(2,1)+C(3,1)+\cdots+C(12,1) = C(\_\_\_,\_\_\_)$ by row-sum pattern

---

Total $= C(2,2)+C(3,2)+C(\_\_\_,\_\_\_)+\cdots+C(\_\_\_,\_\_\_) \quad = C(\_\_\_,\_\_\_)$ by row-sum pattern

$= \_\_\_\_\_\_\_\_\_\_$ gifts

Thanks to ibn Mun'im's shortcut, the entire calculation can be done with a single C(n,r) !

If all of these gifts had been given at the rate of only one per day, instead of over a period of twelve days, how long would the gift-giving have taken?

____

Part of a printed fabric that shows the gifts given in the song "The Twelve Days of Christmas."

Alexander Henry Fabrics, Inc., Burbank, CA

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Exercise 14. Now let's use the row-sum pattern to discover a **column-sum pattern** in the arithmetical triangle.

a) Copy your answers from Exercises 7cd below:

$C(8,5) =$ _______+ _______+ _______+ _______

$C(7,5) =$ _______+ _______+ _______

b) Subtract the bottom equation from the top equation:

$C(\_\_,\_\_) - C(\_\_,\_\_) = C(\_\_,\_\_)$

c) Transpose one term so as to write the equation in part (b) as a sum rather than a difference (in the same way that an equation like, say, 19 – 5 = 14, can be rewritten as 5 + 14 = 19):

$C(\_\_,\_\_) + C(\_\_,\_\_) = C(\_\_,\_\_)$

d) Calculate all 3 terms to make sure this is correct:

_______+ _______ = _______

e) Look at the arithmetical triangle on page 6, and notice that the 2 terms you are adding stand one above the other in the same column, while the sum is immediately to the right in the next column.

Does this column-sum pattern seem to hold throughout the triangle?__________

Write one other example of the column-sum pattern from the triangle:

_______+ _______ = _______

Exercise 15. Use the column-sum pattern to fill in the next column of the arithmetical triangle:

For a tassel of 11 colors											
For a tassel of 10 colors										1	
For a tassel of 9 colors									1	9	
For a tassel of 8 colors								1	8	36	
For a tassel of 7 colors							1	7	28	84	
For a tassel of 6 colors						1	6	21	56	126	
For a tassel of 5 colors					1	5	15	35	70	126	
For a tassel of 4 colors				1	4	10	20	35	56	84	120
For a tassel of 3 colors			1	3	6	10	15	21	28	36	
For a tassel of 2 colors		1	2	3	4	5	6	7	8	9	
For a tassel of 1 color	1	1	1	1	1	1	1	1	1	1	
	using color #1	using color #2 (& earlier)	using color #3 (& earlier)	using color #4 (& earlier)	using color #5 (& earlier)	using color #6 (& earlier)	using color #7 (& earlier)	using color #8 (& earlier)	using color #9 (& earlier)	using color #10 (& earlier)	using color #11 (& earlier)

← 36 + 84 = 120

Exercise 16. Let's note other examples of the column-sum pattern:

exs. $35 + 21 = 56 \rightarrow C(7,4) + C(7,5) = C(8,5)$
$5 + 10 = 15 \rightarrow C(5,1) + C(5,2) = C(6,2)$

a) $C(6,2) + C(6,3) = C(7, \underline{\quad})$
b) $C(8,5) + C(8,6) = C(\underline{\quad}, \underline{\quad})$
c) $C(n,r) + C(n,r+1) = C(\underline{\qquad}, \underline{\qquad})$

Fill in the blanks.

Exercise 17. Finally, let's confirm the column-sum pattern by interpreting the numbers as color combinations.

Suppose that only 7 colors are available, and we want to choose 3 of them. We can either choose color 7, and any 2 of the first 6 colors; or else *not* choose color 7, but any 3 of the first 6 colors. In symbols:

$$C(7,3) = C(\underline{\quad}, \underline{\quad}) + C(\underline{\quad}, \underline{\quad})$$ (fill in the blanks)

↑ No. of ways to choose 3 of 7 colors
↑ No. of ways to choose 2 of the first 6 colors
↑ No. of ways to choose 3 of the first 6 colors

Does this result agree with your answer to Exercise 16a above?_______