Michel Rolle and His Method of Cascades

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Introduction Almost every college calculus course exposes its students to Rolle's Theorem. Hardly ever though, are the students exposed to who Rolle was and how Rolle's Theorem came about. In this paper we discuss Michel Rolle and his Method of Cascades which ultimately led to the theorem that bears his name. Also we discuss his other contributions to mathematics and his critique of the infinitesimal calculus.

Biographical Sketch Michel Rolle, a Frenchmen, was born on April 21, 1652 in the small town Ambert, Basse-Auvergne, France. He was born to a shoemaker and had very little formal education, receiving only some elementary schooling. After his elementary schooling he became a renowned self-educated man, making many contributions to the field of mathematics [3].

During the earlier years of his life he worked as a notary and an assistant to several of the attorneys in and around his hometown. Then in 1675, at the age of 23, Rolle moved to Paris. In Paris he worked as a scribe and an expert arithmetic. Not long after arriving in Paris he married and had children. Unable to support his family, he sought means for a higher income. His self-education in the study of higher mathematics gave him the break he was looking for [3].

Seven years after moving to Paris, Rolle provided a solution to a problem in which had been publicly posed by Jacques Ozanam. This problem was published in the August 31, 1682 edition of the *Journal des savan*. The problem Rolle solved was stated by Ozanam in this fashion *"Find four numbers the difference of any two being a perfect square, in addition the sum of the first three numbers being a perfect square."* Ozanam believed that the smallest of the four numbers that would satisfy these properties would have at least 50 digits. Rolle found four numbers, all satisfying the conditions Ozanam posed, containing only seven digits in each of the four numbers[3].

Letting x, y, w, and z be the four numbers satisfying all the conditions proposed by Ozanam with $x < y < w < z$, we have the following seven equations: $z - w = a^2$, $w - y = b^2$, $y - x = c^2$, $z - y = d^2$, $z - x = e^2$, $w - x = f^2$, and $x + y + w = g^2$, where a, b, c, d, e, f, and g are integers . From these equations, we can see that finding these four numbers is very difficult.

By solving this problem, Rolle gained a mathematical reputation. Rolle was immediately sought after by one of the highest officials of the French government. At the time Rolle solved the daunting task posed by Ozanam of finding the four numbers, John-Baptiste Colbert was the controller general of finance and secretary of state for the navy under King Louis XIV. Colbert rewarded Rolle for his achievement by arranging a pension for Rolle, and thus giving him financial security. Rolle, now not having to worry with finances and wondering whether he would be able to provide for his wife and children, could then pursue further studies.

Rolle also received another big break by solving Ozanam's problem. The French Secretary of State for War Marquis do Louvois' hired Rolle to tutor one of his sons. Louvois' then arranged Rolle to take an administrative post in the Ministry of War. Not liking his work in this position, Rolle soon resigned. However, that wasn't the end of Louvois' influence on Rolle. After becoming greatly impressed with his pedagogic and mathematical skills, Louvois' vouched for Rolle and had an incredible influence on Rolle being elected to the *Acadmie Royale des Sciences* in 1685. Rolle, now a member of a very select society, was able to study mathematics in a much more devoted way than ever before in his life [3].

Rolle's contributions cover a wide range of mathematical ideas. One of his main areas of study was that of Diophantine analysis. Diophantine analysis is the study of diophantine equations(any equation in one or more unknowns) and their solutions in the integers(or rationals)[1]. Rolle's first break in mathematics can actually be attributed to his study in Diophantine analysis.

Rolle also worked in the field of algebra. His most famous work, *Traite d'algebre*, was published in 1690. In this work Rolle was the first to use the symbol $\sqrt[n]{x}$ as the notation for the nth root of x. After using this notation, it became the standard notation for nth roots and is still used today. In his 1691 work *Demonstration d'une Methode pour resoudre les Egalitez de tous les degrez*, or simply his *Demonstration*, Rolle broke with Cartesian techniques. He adopted the notion that if $a > b$, then $-b > -a$. This was not common practice at the time. Also, Rolle used two parallel lines, one on top of the other, to denote "equals." It seems so common that the symbol " $=$ " is used as our equals sign. Rolle didn't invent the symbol we now use today, but rather some may say popularized the notation first invented by Robert Recorde[4]. Recorde stated, "to avoid the tedious repetition of these words-is equal to-I will set as I do often in work use, a pair of parallels, or gemow [twin] lines of one length, thus =, because no two things can be more equal." At the time he published his *Demonstration*, it was very uncommon to use " $=$ " to represent "is equal to" as Descartes was even using a different notation for the same idea. Rolle also published a work based on the solutions of indeterminate equations in 1699 called *Methode pour resoudre les equations indetermines de l'algebre*. In *Trait d'algbre* he also used the Euclidean algorithm to find the greatest common divisor of two polynomials and to solve linear Diophantine equations. See [3] for more details on Rolle and his mathematical contributions.

Method of Cascades However, the most important aspect of Rolle's *Traite d'algebre* is his method of cascades [8]. In this section we will follow the development of [8]. Rolle's Method of Cascades may in fact be his most important contribution to mathematics. This method ultimately lead to the discovery of the calculus theorem that now bears his name, Rolle's Theorem. Rolle's Method of Cascades is a process by which one can find the general solution of numerical equations of the form

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n+1}x^{n+1} + a_n = 0.$$

This method has had a monumental impact on the history of mathematics. Some principles of calculus as well as theory of equations can be traced back to Rolle's method of cascades. This same method amplified the concepts of limits of roots of equations. Rolle didn't prove his method of cascades until a year later in 1691 in *Demonstration*.

We will consider the polynomial

$$f(x) = x^4 + 5x^3 - 25x^2 - 65x + 84 = 0$$

in order to explain the Method of Cascades.

Step 1 We must transform the equation so that its terms alternate between positive and negative. Rolle did this in order to ensure all its roots are positive, as stated in Descartes' Rule of Signs and the Fundamental Theorem of Alebra [10]. First, we will define three terms Rolle used.

1. Rolle defined the "grand hypothese" as the value for which no roots of a polynomial can exceed. He found this number using the formula $(a/c+1)$ where a is the absolute value of the largest negative coefficient and c is the coefficient of the highest degree term. Rolle states this is true of polynomials in general.

2. The "petite hypothese" is the number that is less than every root.

3. The "hypotheses moyennes" are the intermediate bounds, the roots of the cascade.

In order to transform the equation Rolle used a substitution for x. He let

$$x = (\frac{a}{c} + 1) - y$$

where a and c are defined above. In $f(x)$ we see that $a = |-65| = 65$ and $c = 1$ which yields

$$x = (\frac{65}{1} + 1) - y = 66 - y.$$

Substituting $66 - y$ into $f(x)$ we see that

$$\begin{aligned} f(66-y) &= (66-y)^4 + 5(66-y)^3 - 25(66-y)^2 - 65(66-y) + 84 \\ &= y^4 - 269y^3 + 27101y^2 - 1211959y + 20299110. \end{aligned}$$

For simplification purposes, let $f(66-y) = g(y)$. Thus since $g(y)$ has all positive roots, Rolle was able to claim that 0 is the petite hypothese of $g(y)$. We can also find the grand hypothese of $f(x)$ and $g(y)$ using $(a/c+1)$ as defined above. The grand hypothese of $f(x)$ is 66, and all real roots are less than or equal to 66. The grand hypothese of $g(y)$ is 1211960 and no roots of $g(y)$ cannot exceed 1211960.

Step 2 Now we must form the cascades as Rolle did. After transforming $f(x)$ into $g(y)$ we must apply the following method as described by Rolle.

a. *Multiply every term of the equation by the exponent of the unknown variable and divide by the unknown.*

b. *Do the same with the second equation formed from the first. Continue until an equation of the first degree is obtained.*

Each of these equations is what Rolle called a cascade. However, forming "cascades" is equivalent to taking derivatives in succession until we reach an equation of the first degree. Note Rolle did not call these different equations derivatives, but cascades. In this paper we will use these words interchangably. Observe the following about $g(y)$ and its cascades denoted using derivative notation:

$$\begin{aligned}
g(y) &= y^4 - 269y^3 + 27101y^2 - 1211959y + 20299110 = 0 \\
g'(y) &= 4y^3 - 807y^2 + 54202y - 1211959 = 0 \\
g''(y) &= 12y^2 - 1614y + 54202 = 0 \\
g^{(3)}(y) &= 24y - 1614 = 0.
\end{aligned}$$

Step 3 The third and final step of Rolle's cascades is actually solving the equation $g(y) = 0$. In his *Traite d'algebre* Rolle stated that the roots of an equation are seperated by the roots of its cascade. This means that if an equation's cascade has roots of say a and b with $a < b$, then the root(s) of the equation lie between a and b. He then proved this statement in his *Demonstration.* A proof is given on page 6 of this paper. Keep in mind, all of the cascades after step 1, will have positive roots only. Thus each cascade has a petite hypothese of 0, and we can find the grand hypothese. Thus if bounds on the roots of $g'(y)$ can be found, then the roots of $g(y)$ can be found. Likewise, if bounds on the roots of $g''(y)$ can be found, we can find bounds on the roots of $g'(y)$, as well as for $g^{(3)}(y)$ and $g''(y)$.

Since we can easily find the roots of $g^{(3)}(y)$, we can proceed to find bounds for the roots of each of the preceding equations, and ultimately we can find the roots of $g(y)$ and our original equation $f(x)$ by using Rolle's method of cascades. We have yet to find the actual roots of $g(y)$, but through some extensive calculations this can be done.

We will pick up with the calculations at $g'(y)$. The roots of $g'(y)$ are approximately 63, 67, and 71. Thus the petite hypothese of $g(y)$ is 0, the grand hypothese of $g(y)$ is 1211960, and the hypotheses moyennes are 63, 67, and 71. Then the roots of $g(y)$ live in the intervals $(0, 63)$, $(63, 67)$, $(67, 71)$, and $(71, 1211960)$ and each interval contains exactly one root. Now we compute the values of $g(y)$ for $y = 0$ and $y = 63$. Note $g(0) = 20299110$ and $g(63) = -120$ (opposite in sign). Next try the mean of 0 and 63 rounded to a single digit, 31. Note $g(31) = 1682184$ (still opposite in sign). Continue with $y = 47$, note $g(9) = 154440$ (still opposite in sign). Now try $y = 55$, note $g(55) = 17640$ (still opposite in sign). Continue with $y = 59$, note $g(59) = 176$ (still opposite in sign). The last two means for this pair of hypotheses is $y = 61$ and $y = 62$. Note $g(61) = 384$ and $g(62) = 0$. Thus $y = 62$ is the root we sought and the only one on $(0, 63)$.

Next we use the pair of hypotheses 63 and 67. Note $g(63) = -120$ and $g(67) = 120$ (opposite in sign). Now $g(65) = 0$. Thus 65 is another root. Then we use the pair of hypotheses 67 and 71. Note $g(67) = 120$ and $g(71) = -216$ (opposite in sign). The mean of 67 and 71 is 69, and $g(69) = 0$. Thus 69 is another root. The last pair of hypotheses we would use are 71 and 1211960. This would require using the trial hypotheses $1211960, 606015, 303043, 151557, 75814, 37942, 19006, 9538, 4804, 2437, 1254, 662, 366, 218, 144, 107, 89, 80, 75$, and 73 arriving at $g(73) = 0$, the fourth and final root.

Thus the roots of $g(y)$ as found by Rolle's method of cascades are $y = 62, y = 65, y = 69$, and $y = 73$. We can then easily find the roots of $f(x)$ by substituting the roots of $g(y)$ into the substitution we made earlier, $x = 66-y$. The roots of $f(x)$ are then $x = 4, x = 1, x = -3$, and $x = -7$.

Rolle's Method of Cascades can also be used to approximate roots of polynomials that do not have integer solutions. Take for example the polynomial

$$f(x) = x^3 + 3x^2 - 2x - 6 = 0.$$

Now $x = 7 - y$ and

$$f(7 - y) = g(y) = -y^3 + 24y^2 - 187y + 470 = 0.$$

The cascades are as follows

$$\begin{aligned} g(y) &= -y^3 + 24y^2 - 187y + 470 = 0 \\ g'(y) &= -3y^2 + 48y - 187 = 0 \\ g''(y) &= -6y + 48 = 0. \end{aligned}$$

Now we can proceed to use Rolle's Method of Cascades to arrive at approximate solutions for $g(y)$ and $f(x)$. The approximate roots of $g'(y)$ are 6 and 9. Now we will not round the mean to a single digit as we did earlier. Instead we will carry out the decimals to approximate the roots to any degree we wish. The petite hypothese for $g(y)$ is 0 while the hypotheses moyennes are 6 and 9. The grand hypothese is 188. To show that we can approximate roots to any degree we will use the trail hypotheses 6 and 9 only. Note that $g(6) = -4$ and $g(9) = 2$(opposite in sign). Take the mean of 6 and 9, yielding 7.5. Now $g(7.5) = -4.375$ (still opposite in sign from $g(9)$. Take the mean of 7.5 and 9 this time. This mean is 8.25 and $g(8.25) = -.765625$. We see that we are getting closer to the actual root that lies between 6 and 9. Continuing in like fashion we can get as close to the actual root which is $7+\sqrt{2} \approx 8.414213562$ as we would like. Just to emphasize this, let's do one more iteration. The mean of 8.25 and 9 is 8.625. Note $g(8.625) \approx .88086$. Now it is not opposite in sign from $g(9)$. Though it is opposite in sign from $g(8.25)$. Now take the mean of 8.625 and 8.25 which is 8.4375 and $g(8.4375) \approx 0.10376$. We see in each iteration we are narrowing the margin between the actual value $7+\sqrt{2} \approx 8.414213562$ and our approximating values. We can continue to make more and more iterations and become more and more accurate with each one. At each iteration we can use $x = 7 - y$ to get closer and closer to the actual roots of our original equation $f(x)$. Note that $7+\sqrt{2} \approx 8.414213562$ corresponds to the root $-\sqrt{2}$ of $f(x)$.

Rolle's Theorem As we already know, Rolle's method of cascades lead to the discovery of the result that is now named after him. It wasn't until 1846 that this result beared the name "Rolle's Theorem". Giusto Bellavitis was responsible for naming the theorem after the man who discoverd it [3]. Rolle included this result in his 1691 *Demonstration.* Rolle's Theorem as stated in many calculus textbooks is as follows:

Rolle's Theorem Let$f$ be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) with $f(a) = f(b)$. Then there is a number c in (a, b) such that $f'(c) = 0$ [9].

The proof of this statement usually follows in the calculus text. See [9] for a typical proof. Calculus texts then go on to use Rolle's Theorem to arrive at and prove the Mean Value Theorem.

Rolle used the following statement in his method of cascades which is equivalent to his theorem. *If g is a polynomial and $g'(a) = g'(b) = 0$, and there is no number c, $a < c < b$ such that $g'(c) = 0$, then g has at most one real root on (a, b).*

Proof. Note that g is a polynomial (since that is what Rolle worked with) and g is both continuous and differentiable with $g'(a) = g'(b) = 0$ and there is no number c in (a, b) such that $g'(c) = 0$. Assume that g has two real roots on (a, b). This means that there is a number x_1 and x_2 in (a,b) with $x_1 < x_2$ such that $g(x_1) = g(x_2) = 0$. According to Rolle's Theorem there is a number x_3 between x_1 and x_2 such that $g'(x_3) = 0$. However this contradicts that there is no number c in (a, b) such that $g'(c) = 0$. Thus the statement is proven. □

The link between Rolle's Theorem and his Method of Cascades is better understood with a graph. (See figure 1).

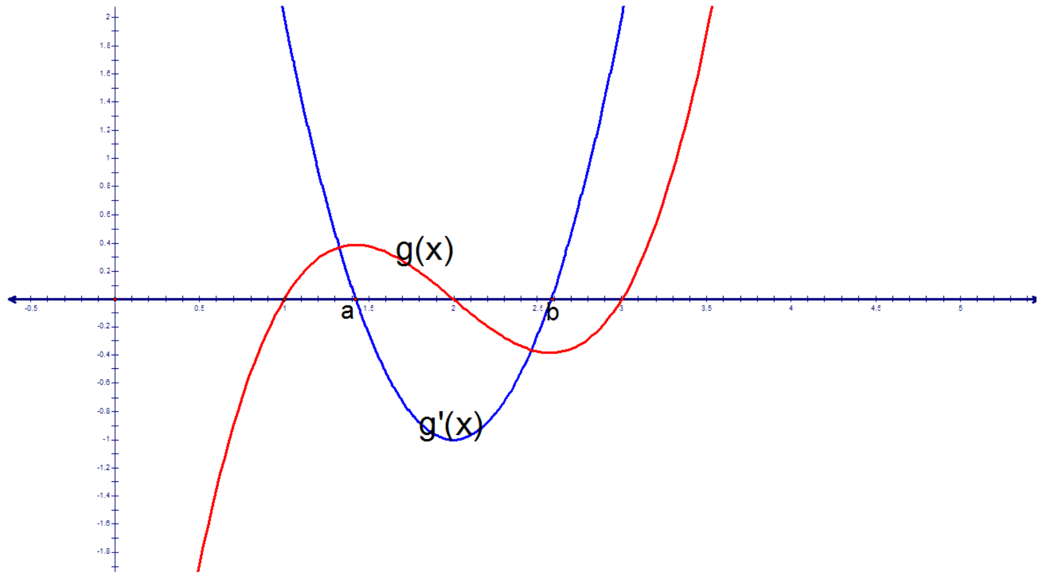

Figure 1

We see that the function and its derivative are pictured. Between each root of the derivative we see at most one real root of the function. It is this idea that Rolle uses in his Method of Cascades. If there were more than one real root between the points where the derivative is zero, the original function would have to acheive a maximum or minimum between the two points, thus creating another point where the derivative is zero. However, upon further observation it is this same idea that appears in Rolle's Theorem, that where $f(a) = f(b)$ there is some number c between a and b such that $f'(c) = 0$.

Extensions of Rolle's Theorem After studying Rolle's Theorem one might wonder if this theorem can be extended into complex analysis (calculus over the complex numbers) or to multivariate calculus over the real numbers. In fact, since the discovery of this theorem attempts to extend this theorem have been more than common. A complex version of Rolle's Theorem was proven in 1992. See [2] for the discussion of the complex version. A multivariate version was also proven in 1980. See [7] for the discussion of the multivariate version.

Conclusion It may be reasonable to think that Rolle was a developer in the infinitesimal calculus. However this is not accurate. Most calculus textbooks across college campuses have a note in the margin beside Rolle's Theorem. One such text says this, "French Mathematician Michel Rolle first published the theorem that bears his name in 1691. Before this time, however, Rolle was one of the most vocal critics of calculus, stating that it gave erroneous results and was based on unsound reasoning"[5]. Rolle's first outlashes at the infinitesimalist came in 1700 and stayed with the French Academy of Sciences. He fought mostly with Pierre Varignon who defended the infinitesimal calculus. At the end of 1701 the Academy silenced Rolle and Varignon. The second part of Rolle's outlashes came in 1702 when Rolle published articles in the *Journal des Scavans*. In these articles Rolle never mentions Varignon, but the articles are a clear challenge to the infinitesimalists. Rolle attacked again in 1703 and 1704. And it was then that Fontenelle joined the battle and also defended the calculus.

However Rolle later retracted his statements after being asked in 1706 to conform better to the regulations of the Academy. It is believed though that Rolle never convinced himself of the soundness of the infinitesimal calculus. In 1708 Varignon wrote to Liebniz and noted that Rolle was still making adverse comments. Even though Rolle's intense scrutiny of the infinitesimals came up empty handed, it was very beneficial. A statement that sums up this idea is best described by Jean Itard when he wrote "Rolle was a skillful algebraist who broke with Cartesian techniques; and his opposition to infinitesimal methods, in the final analysis, was beneficial" [6]. Without Rolle's scrutiny on this concept, calculus may never have been developed as rigrous as it was. For more on Rolle's critique of the infinitesimalists see [6].

As we have seen, Rolle, usually only heard of when discussing the theorem he discoverd through his method of cascades made several significant contributions to the study of mathematics. His work in Diophantine analysis had a big impact on his further study of mathematics. His work with the theory of equations also lead to his monumental impact on mathematics with his method of cascades. Even Rolle's opposition and critiques of calculus helped shape mathematics.

Sadly, at the age of fifty-six years old in 1708 Rolle suffered a stroke. He never fully recovered from this stroke and made no more mathematical contributions after this incident. He did recover his health almost fully, but his mental capacity was never the same afterwards. Then eleven years later in 1719 Rolle suffered another stroke, this time fatal [3].

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