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A π-less Buffon's Needle Problem

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In 1733 the French naturalist Georges Louis Leclerc, Comte de Buffon, posed and solved the following problem in geometric probability: when a needle of length L is dropped onto a wooden floor constructed from boards of width D (where $D \geq L$), what is the probability that it will lie across a seam between two boards? Buffon determined that the probability is $2L/D\pi$. His proof of the now-famous *Buffon's needle problem* appeared in print 44 years later [**5**].

The solution to this problem is straightforward, requiring only the integral of a trigonometric function, and is accessible to students in an integral calculus course (a solution without integration can be found in [**9**, §1.1]). In 1812 Laplace noticed that it is possible to approximate π by tallying repeated drops of the needle (the most remarkable and most suspect example being Mario Lazzarini's 1901 approximation of π to six decimal places after only 3408 tosses [**10, 11, 4**]). This means of estimating π is now a classic application of the Monte Carlo method.

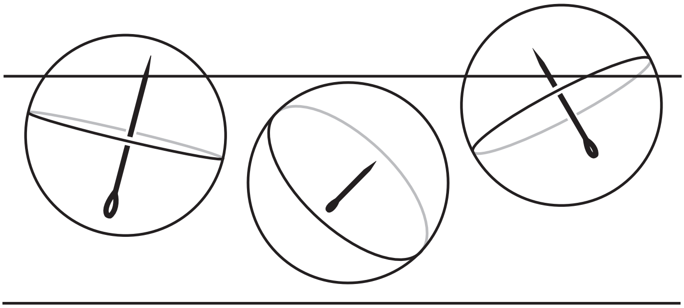

Figure 1

There have been numerous extensions and variations of the Buffon's needle problem (see, e.g. [**1, 6, 7, 8, 10, 12, 13, 14**]). In this note we propose another variation. Suppose that a needle of length L is pushed completely into a clear rubber ball of diameter L. If the ball is dropped onto a wooden floor with slats of width D ($D \geq L$), what is the probability that the needle will lie above a seam in the floor? (This is equal to the probability that a randomly-placed needle in $\mathbb{R}^3$ intersects the set of planes $\{x = kD : k \in \mathbb{Z}\}$.) We call this the *Buffon's ball problem*. We will see that unlike the classical case, π does not appear in the probability, and when $L = D$ the probability is unexpectedly simple.

The exact location of the needle depends on the location of the ball in the plane and on the orientation of the needle in the ball. Thus, it corresponds to a point in the space $\mathbb{R}^2 \times S$, where S is the sphere of diameter L. Taking advantage of symmetries we can reduce the configuration space to $X = [0, D/2] \times S$. That is, we need only know the distance between the base of the ball and the nearest seam (we assume that the boards run east-west) and the orientation of the needle inside the ball. When we say the ball is dropped at random, we mean that the possible configurations are uniformly distributed in X.

For now, fix y, the distance between the base of the ball and the nearest seam. We wish to compute $P(y)$, the probability that the needle crosses a seam for this value of y. Let R_y denote the region of the sphere corresponding to tip locations that yield a crossing and let $A(y)$ denote the area of R_y. Then,

$$P(y) = \frac{A(y)}{\text{area of sphere}} = \frac{A(y)}{L^2\pi}.$$

When $y > L/2$ the needle cannot cross the seam, so $R_y = \emptyset$, $A(y) = 0$, and $P(y) = 0$. When $0 \le y \le L/2$, R_y consists of two identical spherical caps centered about the north and south poles (as in FIGURE 2)—one cap corresponds to the tip being north of the seam and the other to the eye being north of the seam.

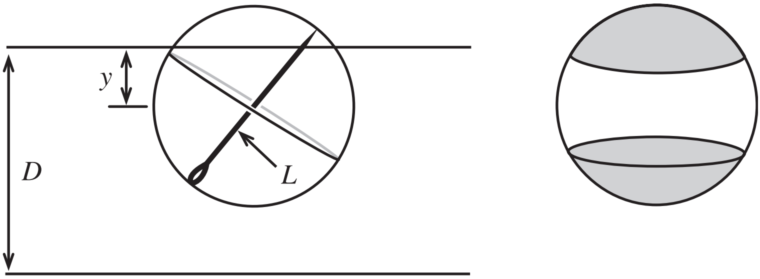

Figure 2

To compute the areas of these spherical caps we turn to Archimedes. In his work *On the Sphere and Cylinder, Book I* [**3**, Proposition 43, p. 53] he proved the remarkable theorem that the area of a spherical cap be expressed in terms of only the slant height of the cap, r (see FIGURE 3). Just like a circle in the plane, the area of the cap is πr^2.

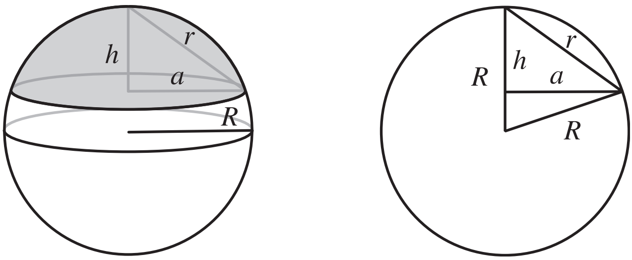

Figure 3

We would like the area of the cap in terms of the radius of the sphere, R, and the height of the cap, h. Taking a to be the radius of the base of the cap, we have

$$a^2 + h^2 = r^2 \quad \text{and} \quad (R-h)^2 + a^2 = R^2.$$

Eliminating a we obtain $r^2 = 2Rh$. Thus, the surface area of the spherical cap is $2\pi Rh$.

Since our spherical caps have height $L/2 - y$, the area of R_y is

$$A(y) = 2\left(2\pi \cdot \frac{L}{2} \cdot \left(\frac{L}{2} - y\right)\right) = L^2\pi\left(1 - \frac{2y}{L}\right).$$

Thus, for the fixed value of y ($0 \le y \le L/2$), the probability that the needle crosses the seam is

$$P(y) = \frac{L^2\pi(1 - 2y/L)}{L^2\pi} = 1 - \frac{2y}{L}.$$

Also note that $P(y) = 0$ for $L/2 \le y \le D/2$.

Finally we are able to solve the Buffon's ball problem. Since every possibility $0 \le y \le D/2$ is equally likely, the probability that the needle will lie above a seam is

$$\frac{2}{D}\int_0^{D/2} P(y)\,dy = \frac{2}{D}\int_0^{L/2}\left(1 - \frac{2y}{L}\right)dy,$$

which, by elementary geometry (see FIGURE 4), equals $L/2D$.

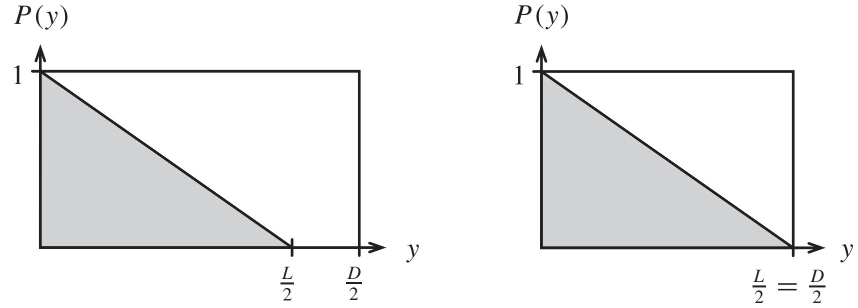

Figure 4

Notice that unlike the classical Buffon's needle problem, π makes no appearance in the probability. Moreover, as is easily seen in FIGURE 4, when $D = L$ the probability of the needle crossing a seam is the same as a coin toss!

Although we justified these surprising conclusions mathematically, it would be nice to gain an intuitive understanding of why they hold. To do so, we return to the formula for the area of a spherical cap, $A = 2\pi Rh$. Using this formula it is easy to show that when a sphere and its circumscribing cylinder are sliced by two planes perpendicular to the axis of the cylinder (see FIGURE 5) they produce slices of equal lateral surface area (this result can be found in [**2**]). Viewed another way, when a sphere is cut by two

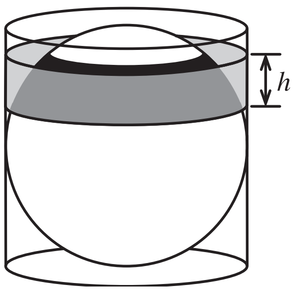

Figure 5

parallel planes, the area bounded by the planes depends only on their separation and not on their orientation.

In the context of the Buffon's ball problem this uniformity implies that the lengths of all projections of the needle onto the north-south axis are equally likely! So, the expected length of the projection is $L/2$. Since the distribution is uniform, the probability of a crossing is $(L/2)/D = L/2D$.

Surveying the literature we find that the classical Buffon's needle problem can be extended in several directions. Of particular interest are the generalizations found in Ramaley's paper [**13**]. First, if the needle is longer than the width of the boards ($L > D$), then it may cross more than one seam when it falls on the floor. In this case the expression found by Buffon, $2L/D\pi$, is the expected value for the number of line-crossings. In fact, Ramaley shows that this interpretation holds even if the needle is not straight, but is a polygonal curve of length L, or, in the limiting case, a curve of length L. That is, if a piece of string of length L is tossed on the floor then $2L/D\pi$ is the expected number of line-crossings! He called this the *Buffon's noodle problem*.

Examining Ramaley's arguments we see that they apply to our situation with no modification except the replacement of Buffon's probability with ours. So, if a needle of length L is placed in a ball of diameter L and is dropped onto a wooden floor with boards of width D, then the expected number of seams that the needle will cross is $L/2D$. Likewise, if a piece of string of length L is suspended inside this same ball so that it cannot move and it is dropped on the same floor, then $L/2D$ is the expected number of seams it will cross (FIGURE 6). Viewed in another way, this says that on average a curve of length L in $\mathbb{R}^3$ will intersect the set of planes $\{x = kD : k \in \mathbb{Z}\}$ in $L/2D$ points.

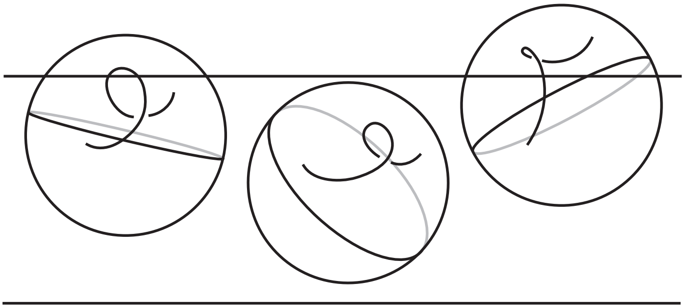

Figure 6

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Conditions Equivalent to the Existence of Odd Perfect Numbers

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The abundancy index $I(n)$ of a positive integer n is defined to be the ratio $I(n) = \sigma(n)/n$, where $\sigma(n) = \sum_{d|n} d$. This index is a useful tool in determining whether a number is deficient, abundant, or perfect. In particular, n is *deficient* if $I(n) < 2$, it is *abundant* if $I(n) > 2$, and *perfect* if $I(n) = 2$. Some of the oldest open problems in mathematics relate to the abundancy of a number. Are there infinitely many perfect numbers? Does there exist an odd perfect number? Here are just two questions that were posed by the Greeks over two thousand years ago, and yet they remain unanswered today.

In recent years, this MAGAZINE has published several interesting articles examining the abundancy index of a number [**1, 3, 4**]. In [**1**], R. Laatsch provided a comprehensive summary of what is known about the abundancy index, including a proof that the image of $I(n)$ is dense in the interval $(1, \infty)$. He also posed several interesting questions, one of which was: *Is every rational number $q > 1$ the abundancy index of some integer?* In [**4**], P. A. Weiner answered Laatsch's question in the negative by providing an infinite family of rational numbers in $(1, \infty)$ which fail to be an abundancy index of any integer. Even more interesting, Weiner proved that the set of rationals in $(1, \infty)$ not in the range of $I(n)$ is actually dense in $(1, \infty)$. Finally, Weiner proved the following result:

THEOREM. (WEINER, 2000) *If $I(n) = \frac{5}{3}$ for some n, then $5n$ is an odd perfect number.*

In [**3**], R. F. Ryan then generalized this theorem of Weiner by proving the following:

THEOREM. (RYAN, 2003) *If there exists a positive integer n and an odd positive integer m such that $2m - 1$ is a prime not dividing n and*

$$I(n) = \frac{2m-1}{m},$$

then $n(2m - 1)$ is an odd perfect number.

In Theorem 1 below, we generalize Ryan's result further by providing a condition that is actually equivalent to the existence of odd perfect numbers. The generalization