for all integers m. By means of (6) and (7), we obtain that for all integers m, n:

$$f(2^n 3^m) = (2^n 3^m)^2 f(1). \tag{8}$$

By our previous lemma any real numbers in $[0, \infty)$ may be approximated by a sequence in the set $\{2^n 3^m | n, m \text{ integers }\}$ so from (8) and the continuity of f we can conclude that for all x in $\mathbb{R}^+$, $f(x) = kx^2$, with $k = f(1) > 0$ an arbitrary constant. ■

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Monotonic Convergence to *e* via the Arithmetic-Geometric Mean

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Recently, Hansheng Yang and Heng Yang [**3**], by using only the arithmetic-geometric inequality, have proved the monotonicity of the sequences (x_n), (y_n), related to the number e:

$$x_n = \left(1 + \frac{1}{n}\right)^n, \quad y_n = \left(1 + \frac{1}{n}\right)^{n+1} \quad (n = 1, 2, \dots)$$

Such a method probably is an old one and has been applied e.g. in [**1**], or [**2**].

We want to show that the above monotonicities can be proved much easier than in [**3**].

Recall that the arithmetic-geometric inequality says that for $a_1, \dots, a_k > 0$, and

$$G_k = G_k(a_1, \dots, a_k) = \sqrt[k]{a_1 \dots a_k},$$
$$A_k = A_k(a_1, \dots, a_k) = \frac{a_1 + \dots + a_k}{k},$$

we have

$$G_k \le A_k, \tag{1}$$

with equality only when all a_i are equal.

Let $k = n+1$, $a_1 = 1$, and $a_2 = a_3 = \cdots = a_{n+1} = 1 + \frac{1}{n}$. Then

$$G_{n+1} = \left(1 + \frac{1}{n}\right)^{n/n+1} \quad \text{and} \quad A_{n+1} = 1 + \frac{1}{n+1},$$

and raising both sides to the $n+1$ power gives

$$x_n < x_{n+1}. \tag{2}$$

For $k = n+2$, $a_1 = 1$, and $a_2 = a_3 = \cdots = a_{n+2} = 1 - \frac{1}{n+1}$, we have

$$G_{n+2} = \left(1 - \frac{1}{n+1}\right)^{n+1/n+2} \quad \text{and} \quad A_{n+2} = \frac{n+1}{n+2}.$$

Raising both sides to the $n+2$ power and taking reciprocals gives

$$y_{n+1} < y_n. \tag{3}$$

Clearly $y_n - x_n = x_n \cdot \frac{1}{n} > 0$, so $x_n < x_{n+1} < y_{n+1} < y_n < \cdots < y_1 = 4$ for $n > 1$, thus the sequences (x_n), (y_n) are convergent, having the same limit (denoted by e). Clearly, $x_n < e < y_n$, so (2) and (3) give

$$x_n < x_{n+1} < e < y_{n+1} < y_n. \tag{4}$$

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