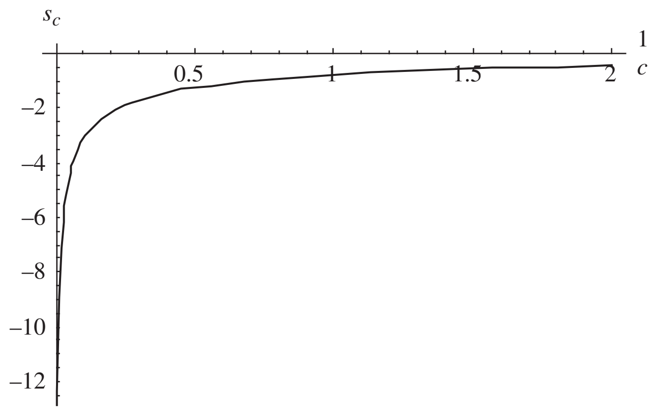

Figure 3. s_c as a function of $\frac{1}{c}$.

There are of course any number of follow-up questions that can be phrased. Here are several for the interested reader.

- What (if anything) does the change in concavity for $c > .5$ mean?
- Which point (if any) on the wall gets the wettest? s_c?, the point closest to the sprayer?, the inflection point?, some other point?
- Least wet?
- What about different wall shapes? Circular? Other conic sections (depending on eccentricity)?
- Is there a good way of expressing the answer for an arbitrarily-shaped wall?
- What if the water slows down in its traverse through the air?
- Take gravity into account.

Reference

1. S. C. Althoen and J. F. Weidner, Related rates and the speed of light, *College Math. J.* **16** (1985) 186–189.

Snapshots of a Rotating Water Stream

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Physics teaches us that trajectories are parabolic if we neglect air resistance and change in altitude. Everyday experience seems to corroborate the theoretical equations. For example, when my neighbor waters a spot on the lawn, the stream from the hose appears to be parabolic, as anticipated. However, when the hose is raised to reach another spot, something happens to momentarily give me pause. I no longer see parabolas while the hose is in transition. Of course, we resolve this mystery by realizing that each drop is on a different parabolic trajectory.

Herein we will describe these new curves, which are snapshots of the stream taken at fixed moments, by expressing each drop's location in terms of the time that has elapsed since its origin. We will investigate only the most elementary case, but the idea is quite adaptable to the more complicated situations involving air resistance or allowing either the angular velocity of the source or the initial speeds of the stream to vary.

Picture a nozzle at the origin rotating counter-clockwise in a vertical plane at constant angular velocity ω with the angle to the horizontal, θ, being 0 at time $t = 0$. Of course, $\theta = \omega t$. At snapshot time s, each drop of the water stream in the air began its flight at an earlier time, t. If we allow the snapshot time to advance, the location of this drop can be thought of as a function of s. The drop has been in the air for time interval $T = s - t$; that is, the time elapsed since its origin equals the difference between the snapshot time and its time of origin. If v represents the nozzle speed, then, in [**1**, pp. 73–74], we see that the coordinates of the drop are given by $x = vT\cos(\omega(s - T))$ and $y = vT\sin(\omega(s - T)) - gT^2/2$, where $T \leq s$. Fortuitously, if we hold photo time s constant, these equations then describe the stream at that moment in terms of parameter T. At airborne time $T = 0$, the drop is at the origin. The larger T becomes, the further back in time the drop left the nozzle, and the farther along the curve the drop is away from the origin.

Those parametric equations indicate that the snapshot curves are not parabolic. In order to picture those curves, we now look at two sequences of photos which allow snapshot time, s, to progress while nozzle speed, v, and angular velocity, ω, remain constant in each Figure. We do not require the nozzle to be at ground level, so that y can be negative. Moreover, in order to limit the number of curves in each Figure, we only graph drops originating from the first three quarters of a revolution of the nozzle. For each curve, newer drops are nearer the origin, and older drops are farther out along the curve. Curves representing snapshots taken at later times originate at larger polar angles.

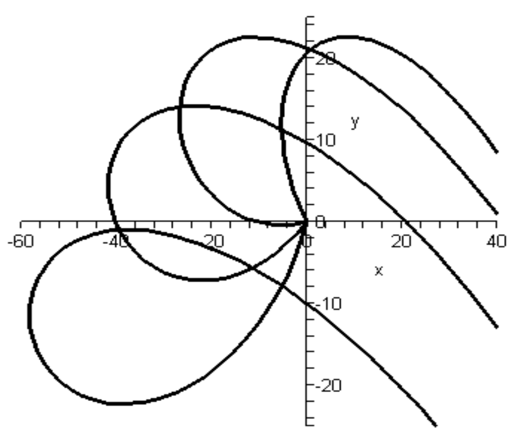

Figure 1.

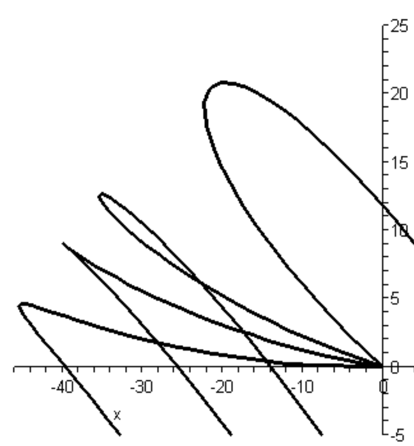

Figure 2.

Figure 1 shows the progression of snapshots if $\omega \geq g/v$. In contrast, when $0 < \omega < g/v$, Figure 2 pictures the progression of snapshots where the loop collapses into a cusp. By solving $x = y = 0$, we learn that, in either case, a loop first appears in the photo at $s = T + \pi/(2\omega)$, where $T = 2v/g$.

We will now see that the relationship between v and ω which allows the cusp to form is indeed $v\omega < g$; that is, the angular velocity must be smaller than g/v. The cusp exists where $x' = y' = 0$. Algebraic manipulation yields that $v\omega < g$ allows the equations to be solved. The result is that the cusp occurs when the elapsed time and snapshot time are respectively $T = v/(g^2 - v^2\omega^2)^{1/2}$ and $s = T + \arccos(-v\omega/g)/\omega$. The coordinates of the cusp are $x = -v^2\omega T/g$ and $y = T^2(g^2 - 2v^2\omega^2)/2g$.

Finally, as an example of a more complicated model to which we can adapt our discussion, we mention the model where air resistance is proportional to velocity. The governing differential equations for x and y are $x'' + rx' = 0$ and $y'' + ry' = -g$ respectively. The independent variable is T, and the initial conditions are $x(0) = y(0) = 0$, $x'(0) = v\cos(\omega t)$ and $y'(0) = v\sin(\omega t)$. Of course, these last two are constants, and t is still equal to $s - T$. After solving, we find that the coordinates of a curve in the photo are given by equations $x = v\cos(\omega(s - T))(1 - e^{-rT})/r$ and $y = (g + rv\sin(\omega(s - T)))(1 - e^{-rT})/r^2 - gT/r$. Figures 3 and 4 illustrate this family of new curves. To facilitate comparison to Figures 1 and 2, we retain their values of v, ω, and s. In all Figures, $v = 40$ and $g = 32$. In Figures 1 and 3, $\omega = 1.0$ and $s = \{2.3, 3.3, 3.8, 4.3\}$. In Figures 2 and 4, $\omega = 0.5$ and $s = \{5.3, 5.9, 6.093, 6.3\}$. For Figures 3 and 4, we use $r = 1$.

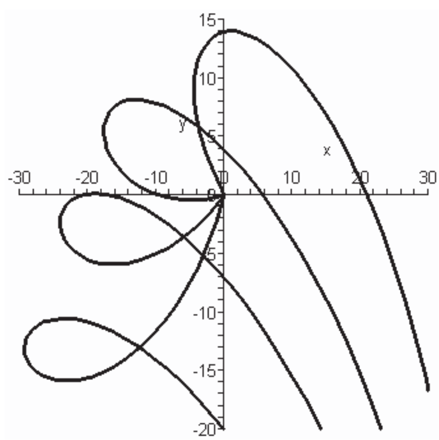

Figure 3.

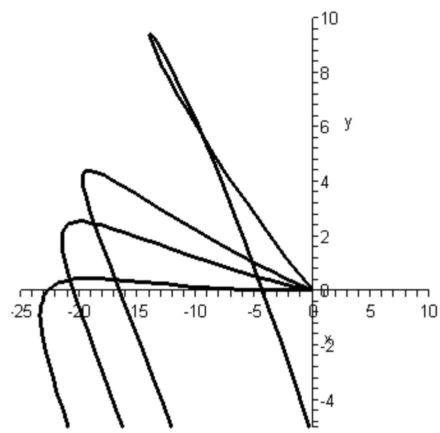

Figure 4.

Reference

1. D. Halliday, R. Resnick, and J. Walker, *Fundamentals of Physics*, 4th ed., Wiley, 1993.

The Computation of Derivatives of Trigonometric Functions via the Fundamental Theorem of Calculus

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Historically, there have been a variety of approaches to computing the derivatives of the sine and cosine functions. The earliest compilation is that of R. Cotes (1682–1716) in his *Harmonia mensurarum*, published by R. Smith in 1722 (see the extensive discussion in Part 2, Chapter 3, § 2, of [**2**]). The first systematic study of these func-