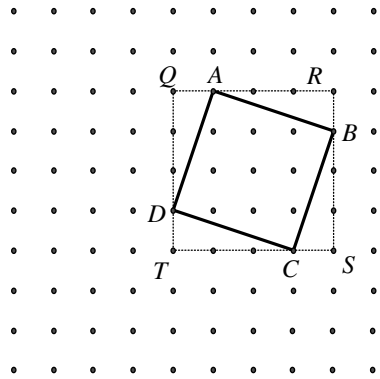

Figure 3.

with sides aligned to the axes of the grid. Each of these squares circumscribes $n-k$ lattice squares, since this is the number of choices of the upper leftmost vertex A of a square $ABCD$ that is inscribed in $QRST$. Altogether, we see there are $\sum_{k=1}^{n-1} k^2(n-k)$ lattice squares in the $n \times n$ lattice.

I'll leave to the reader the challenge of providing combinatorial reasoning that shows the number of squares in an $n \times n$ lattice is also given by $\frac{1}{3}\binom{n}{2}\binom{n+1}{2}$. Neither C. G. Wastun nor I has yet been successful in this effort, so we welcome your ideas.

References

1. Arthur T. Benjamin and Jennifer J. Quinn, *Proofs That Really Count: The Art of Combinatorial Proof*, Mathematical Association of America, 2003.
2. Arthur T. Benjamin and Michael E. Orrison, Two quick combinatorial proofs of $\sum_{k=1}^{n} k^3 = \binom{n+1}{2}^2$, *The College Mathematics Journal* **33** (2002) 407–409.
3. Roger B. Nelsen, *Proofs Without Words II*, Mathematical Association of America, 2000.

Periodic Points for the Tent Function

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The tent function is a piece-wise linear function from the unit interval to itself obtained by letting $f(x) = 2x$ for $0 \le x < 1/2$ and $f(x) = 2 - 2x$ for $1/2 \le x \le 1$. This function is a rich source of examples that can be used to illustrate various concepts in iteration theory (see [**1**]). In particular, the tent function has periodic points of all possible periods. In this note we will investigate some of the properties of these periodic points.

The material in this note is especially suitable as a classroom unit designed to introduce students to some of the basic ideas of iteration theory. This classroom unit can be presented at just about any level since there are essentially no prerequisites other than a familiarity with how base 2 expressions work. It can also serve as a supplement to

the discussion of the tent function in an introductory course dealing with topics such as mathematical chaos.

First some conventions and notation. Throughout this note we will assume that f represents the tent function and (unless otherwise noted) fractions of the form c/d have $0 < c < d$ and $(c, d) = 1$. For each natural number n, we let $M(n) = 2^n - 1$ and $F(n) = 2^n + 1$.

The main result of this note is given by the following theorem.

Theorem.

(1) *x is a periodic point for f if and only if $x = c/d$ where c is even or $x = 0$.*

(2) *Let $x = c/d$ where c is even and let n be the smallest value for which $M(n)$ is a multiple of d. If n is even and $F(n/2)$ is a multiple of d, then x has period $n/2$. In all other cases, x has period n.*

Note that in the above theorem, if d is a prime and n is even, then c/d must have period $n/2$. This follows since $M(n) = F(n/2)M(n/2)$ and by the definition of n, $M(n/2)$ can not be a multiple of d. We also have that if d is a Mersenne prime, say $d = M(p)$, then c/d is periodic of period p for every even c.

Before giving a proof of the theorem it may be useful to give some examples.

For points of period 4 there are two possibilities for $x = c/d$.

1. d divides $M(4)$, but does not divide $M(n)$ for $n < 4$ and does not divide $F(2)$.
2. d divides $F(4)$, but does not divide $M(n)$ for $n < 8$.

For Case 1, we have $M(4) = 3 \cdot 5$. Since 3 divides $M(2)$ and 5 divides $F(2)$, the only fractions which satisfy Case 1 are $2/15$, $4/15$, $8/15$ and $14/15$. So Case 1 gives 4 points of period 4.

For Case 2, $F(4) = 17$ and a quick check shows that 17 does not divide $M(n)$ for $n < 8$. So Case 2 gives 8 points of period 4, namely $2/17, \ldots, 16/17$.

For points of period 5, $M(5) = 31$, so $2/31, \ldots, 30/31$ give 15 points of period 5. $F(5) = 33 = 3 \cdot 11$ and since 11 does not divide $M(n)$ for $n < 10$, we need eliminate only $22/33 = 2/3$ from the fractions of the form $c/33$ with c even. This leaves 15 more values of period 5.

A proof of Part (1) of the theorem is given in [**1**]. We will include a (different) proof of Part (1) in this note as part of the argument used to establish Part (2) of the theorem.

The proof of the theorem will involve dealing with base 2 expressions. To simplify the notation we will suppress the subscript so that $.a_1a_2a_3\ldots_{(2)}$ will be written $.a_1a_2a_3\ldots$.

In base 2 any x with $0 \le x < 1/2$ has the form $.0a_2a_3\ldots$, so $f(x) = 2x = .a_2a_3\ldots$. Similarly, any x with $1/2 \le x \le 1$ has the form $x = .1a_2a_3\ldots$, so $f(x) = 2 - 2x = 1 - .a_2a_3\ldots = .b_2b_3\ldots$ where $b_k = 1 - a_k$ for all k. Thus if $x = .a_1a_2a_3\ldots$, then $f(x) = .a_2a_3\ldots$ for $a_1 = 0$ and $f(x) = .b_2b_3\ldots$ for $a_1 = 1$. A simple induction argument shows that in general the iterates of f have the following property.

$$\text{If } x = .a_1a_2\ldots, \text{ then } f^n(x) = .a_{n+1}a_{n+2}\ldots \text{ provided } a_n = 0$$
$$\text{and } f^n(x) = .b_{n+1}b_{n+2}\ldots \text{ where } b_k = 1 - a_k \text{ provided } a_n = 1.$$

It follows that $f^n(x) = x$ if and only if either

(a) $a_n = 0$ and $a_{k+n} = a_k$ for all k or

(b) $a_n = 1$ and $a_{k+n} = 1 - a_k$ for all k.

In Case (a), x is represented by a repeating base 2 expression with repeating block of length n given by $a_1a_2\ldots a_n$ and in Case (b) x is represented by a repeating expression of length $2n$ given by $a_1a_2\ldots a_nb_1b_2\ldots b_n$ where $b_k = 1 - a_k$.

Note that in each case x is represented by a repeating block that ends in zero.

In general x is represented by a repeating block of length k in base 2, say $x = .a_1\ldots a_ka_1\ldots a_k\ldots$, if and only if $2^kx = a_1\ldots a_k.a_1\ldots a_k\ldots$. Subtracting x from 2^kx and dividing, we have that x must have the form $c/M(k)$ (not necessarily in lowest terms) where c equals the base 2 expression $a_1\ldots a_k$. In particular, $a_k = 0$ if and only if c is even.

It follows that the non-zero periodic points of f are those of the form c/d where c is even.

Conversely, if c is even then c/d must be represented in base 2 by a repeating block that ends in 0; hence c/d must be a periodic point.

To determine the period of c/d, we first note that the length of the smallest base 2 repeating block that represents c/d is equal to r if and only if r is the smallest value for which $M(r)$ is a multiple of d. This follows since as noted above, if x is represented by a (minimal) repeating block of length r, then x is equal to a fraction with denominator equal to $M(r)$. If r is odd, then this value must be the period of c/d, i.e., Case (a) holds. To complete the proof of the theorem we must show that for r even, Case (b) holds if and only if d divides $F(r/2)$.

Assume Case (b) holds for $x = c/d$, i.e., $x = .a_1a_2\ldots a_sb_1b_2\ldots b_s$ where $b_k + a_k = 1$ and $r = 2s$, then,

$$2^sx + x = a_1\ldots a_s.b_1\ldots b_sa_1\ldots a_sb_1\ldots b_s\ldots + .a_1\ldots a_sb_1\ldots b_s\ldots. \qquad (*)$$

The right side of $(*)$ is equal to an integer given in base 2 by $a_1\ldots a_s$ plus the base 2 expression $.111\ldots = 1$. That is, the right side of $(*)$ is an integer. The left side of $(*)$ equals $F(s)\cdot x = F(s)\cdot c/d$. Since $(c, d) = 1$, it follows that d must divide $F(s) = F(r/2)$.

Conversely, assume that d divides $F(r/2)$ where r is the minimal value for which d divides $M(r)$. In this case c/d must be represented by a repeating expression of length r, say $c/d = .a_1\ldots a_ra_1\ldots a_r\ldots$. Letting $s = r/2$, we have

$$F(s)\cdot c/d = F(s)x = 2^sx + x = a_1\ldots a_s.a_{s+1}\ldots a_{2s}\ldots + .a_1\ldots a_s\ldots. \qquad (\#)$$

Since d divides $F(s)$, the left hand side of (#) is an integer. So the right-hand side must also be an integer. Since x is not zero, this is possible only if $a_{k+s} + a_s = 1$ for all k, i.e., Case (b) holds.

Reference

1. D. Gulick, *Encounters with Chaos*, McGraw-Hill, 1992.

Correction from the January 2004 issue: Titu Andreescu has been succeeded by Steven Dunbar of the University of Nebraska as Director of the MAA Mathematics Competition; Titu has joined the IMO Advisory Board.