$$
\begin{aligned}
&= \frac{1}{u} \cdot v - \int v \cdot \left(-\frac{1}{u^2}\right) du \\
&= \frac{v}{u} + \int \frac{v}{u^2}\, du
\end{aligned}
$$

as before.

Secondly, there is the potential only for slight technical advantage in choosing formula (2) over formula (1). An identical integral will need to be computed whether we use (1) or (2). The only difference in the required differentiation and integration occurs in the computation of du versus dU. In our example, for instance, we differentiated $u = x^{1/2}$ rather than $U = x^{-1/2}$.

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Reference

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Estimating Definite Integrals

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Many definite integrals arising in practice can be difficult or impossible to evaluate in finite terms. Series expansions and numerical integration are two standard ways to deal with the situation. Another approach, primitive but often very effective, yields cruder estimates by replacing a nasty integrand with nice functions that majorize or are majorized by it. With luck and skill, the bounds achieved suffice for the task at hand. I was introduced to this method as a grad student instructor over forty years ago, when I had the good fortune to learn some innovative teaching methods from Arthur Mattuck. His supplementary notes for MIT's calculus course included a section on the estimation of definite integrals by an approach barely covered in texts back then. Many first year calculus texts of that era touched on the method in connection with comparison tests for improper integrals, but they seldom did anything with proper integrals.

The situation has improved somewhat in recent years, with prominent texts at least mentioning the basic idea within the chapter introducing the definite integral. Sometimes this is labeled the "Domination Rule" or "Comparison Property". An informal survey shows that most such books offer very few, if any, exercises in the method, usually relatively trivial ones. The texts by Edwards & Penney [**1**] and Stewart [**4**] are exceptional in providing more than a token selection of such problems. Unfortunately, their exercises are duplicated in the early transcendental versions of these two texts, thus making no use of the broader array of available functions.

Here is a sketch of the way I develop this form of estimation in my intermediate calculus course. All integrals are understood to be over a closed, bounded interval $[a, b]$

and all functions assumed to be (Riemann) integrable. I start with the primitive observation that if f is nonnegative and integrable on $[a, b]$ then $\int_a^b f \geq 0$. (This affords an opportunity to remind students of a definition of $\int_a^b f$ and some of its implications.) Next, using the linearity of the integral and the fact that sums and differences of integrable functions are integrable, I infer that for integrable f and g, if $f \leq g$ on $[a, b]$ then $\int_a^b f \leq \int_a^b g$. This last inequality is the key tool. As with any tool, its effectiveness depends on the skill with which it's wielded. For a given function f or g, the trick is to dream up an appropriate comparison function that leads to a useful bound.

Several fundamental inequalities facilitate this effort:

1) $\sin x < x$ for all positive x. (This is easily illustrated with graphics devices, but asking students why it's true can lead to a review of basic methods from a first course in calculus.)
2) $\ln x < x$ and $\ln x < \sqrt{x}$ for all $x > 0$. (Again, easily illustrated and an occasion for pointing to the relevance of earlier methods.)
3) $x^c < x^d$ if $c > d$ and $0 < x < 1$. (As shown below, this simple observation can turn messy integrals into trivial ones.)
4) If f and g are continuous on $[a, b]$ and $f \leq g$ yet $f(c) < g(c)$ for some c in $[a, b]$, then $\int_a^b f < \int_a^b g$. (This can serve as an occasion to review continuity.)
5) If f is integrable on $[a, b]$ then so is $|f|$, and $|\int_a^b f| \leq \int_a^b |f|$. (This continuous analog of the triangle inequality is an easy way to eliminate pesky terms of the form $(-1)^n$.)

Examples.

i) Which of $\int_1^6 \arctan(\sin x)\, dx$, $\int_1^6 \arctan \sqrt{x}\, dx$, $\int_1^6 \arctan x\, dx$ has the greatest value? Which has the least value? Why? This simply relies on the monotone increasing nature of the inverse tangent function, together with the obvious relations among $\sin x$, $\sqrt{x}$, and x over the interval $[1, 6]$.

ii) Similarly, comparing $\int_0^{1/2} \cos x^2\, dx$ with $\int_0^{1/2} \cos \sqrt{x}\, dx$ amounts to combining the inequality $x^2 < \sqrt{x}$ on $(0, 1)$ with the observation that cosine is decreasing on this interval, so $\cos(x^2) > \cos(\sqrt{x})$. Thus the first integral is larger.

iii) Verify that

$$\int_{10}^{15} \frac{t^3}{t^6 + t^2 - 1}\, dt < .003.$$

On $[10, 15]$, $t^2 - 1 > 0$, so deleting the $t^2 - 1$ in the denominator increases the integrand to t^{-3}. Integrating this from 10 to 15 gives as an upper bound:

$$-\frac{1}{2}\left(\frac{1}{15^2} - \frac{1}{10^2}\right) = \frac{1}{360},$$

which is $< .003$.

iv) Verify that

$$\frac{3}{4} < \int_0^1 \frac{1}{1 + t^4}\, dt < \frac{9}{10}.$$

This can be delicate. The lower bound is readily obtained by observing that $1 + t^4 < 1 + t^2$ on $(0, 1)$ and using the inverse tangent. The second inequality appeared on a final examination at our college over thirty years ago, and caused considerable consternation. One route would have been via partial fractions, but our aim was to avoid such extensive computation. Various upper bounds for $1/1 + t^4$ were unsuccessfully proposed among those teaching the course (they would have to grade the test and explain things to curious or unhappy students). Finally my colleague David Armacost suggested showing, instead, that one *minus* the integral is greater than one tenth. This is relatively simple:

$$1 - \int_0^1 \frac{1}{1+t^4}\,dt = \int_0^1 \frac{t^4}{1+t^4}\,dt > \int_0^1 \frac{t^4}{1+1}\,dt = \frac{1}{10}.$$

v) One of the standard examples in calculus involves computing the volume and surface area of the solid of revolution obtained by rotating the graph of $y = 1/x$ about the x-axis for $x \geq 1$. Known as *Gabriel's Horn*, this paradoxical object has an easily computed finite volume, while the surface area is infinite. The area computation,

$$\lim_{A\to\infty} \int_1^A 2\pi \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,dx,$$

is a good exercise in techniques of integration. However, to show the unboundedness of the surface area it's sufficient simply to observe that the integrand is larger than $2\pi\frac{1}{x}$ and that the resulting $2\pi \ln A$ blows up when $A \to \infty$. (Cf. "The Paradox of Gabriel's Horn" in the Varberg, Purcell, Rigdon text [**5**, pp. 418, 419].)

vi) As a final application, we note that without appeal to the theorem allowing termwise integrability of series, it's possible to represent $\arctan(x)$ and $\ln(1 + x)$ as power series. Starting with the fundamental finite geometric sum formula:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \frac{x^{n+1}}{1-x} = \left(\sum_{k=0}^{n} x^k\right) + \text{error term},$$

substitute $x = -t^2$ and $-t$, respectively. Integrating from 0 to x yields $\arctan(x)$ and $\ln(1 + x)$ as finite power series plus integrals of the error term, respectively. It's not difficult to show that as $n \to \infty$, the integral of the error term disappears for $-1 \leq x \leq 1$ in the case of $\arctan(x)$ and for $-1 < x \leq 1$ in the case of $\ln(1 + x)$, yielding the familiar power series for these functions.

For instance, in the case of $\ln(1 + x)$, let $\varepsilon \in (0, 1)$ and suppose $-1 + \varepsilon \leq x \leq 0$. The error term integral is

$$\int_0^x \frac{(-t)^{n+1}}{1+t}\,dt.$$

Because $x \leq 0$, it helps to change variables via $t = -v$, so $0 \leq v \leq -x = |x|$. This facilitates using the continuous analog of the triangle inequality. Thus the magnitude of the error term integral satisfies, for fixed $x \in [-1 + \varepsilon, 0]$,

$$\left|\int_0^x \frac{(-t)^{n+1}}{1+t}\,dt\right| = \left|\int_0^{|x|} \frac{v^{n+1}}{1-v}(-1)\,dv\right| \le \int_0^{|x|} \left|\frac{v^{n+1}}{1-v}\right| dv = \int_0^{|x|} \frac{v^{n+1}}{1-v}\,dv$$
$$\le \int_0^{|x|} \frac{v^{n+1}}{1-|x|}\,dv = \frac{|x|^{n+2}}{(n+2)(1-|x|)}.$$

This shrinks to zero as $n \to \infty$, indeed uniformly for $x \in [-1+\varepsilon, 0]$.

The corresponding result when $x \in [0, 1]$ is similar but easier, because there is no need to avoid an endpoint at which the integrand blows up. For the arctan function it's also easy to show that, for $-1 \le x \le 1$, the error integral goes to zero as $n \to \infty$. This latter analysis has appeared in George Thomas's calculus texts at least since 1960 (see [**2**, pp. 689–690].)

The basic approach used in this sort of work may remind readers of the comparison test for infinite series. However, the series method tends to be more simple in classroom practice, for two reasons. First, the traditional repertory of comparison series is relatively limited (p-series and geometric series, with perhaps also factorial series). Second, in the study of a series the primary goal is to find out whether it converges, and for this the limit comparison test obviates the need to confirm an inequality. When the task turns to bounding or estimating the sum of a convergent series or to assessing the rate of growth of a divergent series, the approaches become more similar.

The above examples are elementary, yet they suggest the flexibility needed in approximations as well as the significant simplification provided by changing our point of view. Students find these problems difficult precisely because they cannot be solved by rote-learned algorithms. As Richard Feynman remarked when discussing skill in making approximations, "This is very difficult to teach because it's an art." [**3**, p. 16]

This topic reinforces the view that calculus is the study of the behavior of functions. More importantly, it shows mathematics as an art, as an endeavor probing relationships among elements in an infinite universe.

Acknowledgment. I am grateful to the referee for alerting me to the presence of this method in contemporary calculus texts.

References

1. C. H. Edwards and D. E. Penney, *Calculus*, 6th ed., Prentice Hall, 2002.
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3. D. and J. Goodstein, " 'The Uncreative Scientist'—Feynman's *Other Lost Lecture*", *Caltech News*, **31**:4 (1997) 12–17.
4. J. Stewart, *Calculus*, 5th ed., Thomson Learning, 2003.
5. D. Varberg, E. J. Purcell, and S. E. Rigdon, *Calculus*, 8th ed., Prentice Hall, 2000.

Added in proof: It has come to the author's attention that there was an earlier Capsule in this *Journal* illustrating the repeated use of the basic comparison property to obtain improved estimates of definite integrals. See W. V. Underhill, Finding bounds for definite integrals, *The College Mathematics Journal* **15** (1984) 426–429.