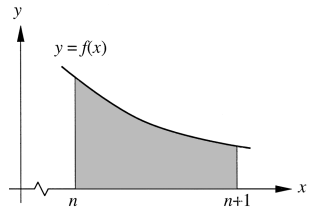
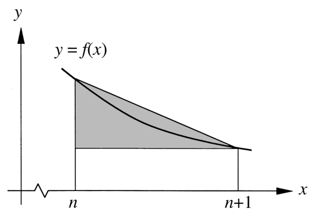
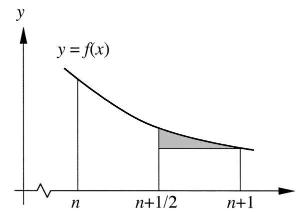


(a) Estimate (1)



(b) Estimate (2)



(c) Estimate (3)

Figure 1.

$f(x) = x^{-4}$ and $\varepsilon = .5 \times 10^{-6}$ yields $n \geq 16$. Using $S_{16} \cong 1.08224917$ (to 8 places) and (3) yields $1.08232300 \leq S \leq 1.08232337$; hence $S \cong 1.082323$, correct to 6 places. To obtain the same accuracy with (2) requires $n \geq 21$, and with (1) requires $n \geq 38$.

Reference

1. R. K. Morley, The remainder in computing by series, *American Mathematical Monthly* **57** (1950), 550–551.

A Modified Discrete SIR Model

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A couple of years ago in a Calculus I course, I introduced my students to the classic Kermack-McKendrick SIR model [**1**] for the spread of an epidemic:

$$
\begin{aligned}
\frac{dS}{dt} &= -aS(t)I(t), \\
\frac{dI}{dt} &= aS(t)I(t) - bI(t), \\
\frac{dR}{dt} &= bI(t),
\end{aligned}
\tag{8}
$$

where $S(t)$, $I(t)$, and $R(t)$ represent, respectively, the number of "susceptibles," "infecteds," and "recovereds" in a population at time t, with time measured in days. The term $aS(t)I(t)$ models an assumption that the rate at which susceptible individuals become infected is proportional to the number of interactions between susceptibles and infecteds, with this number of interactions itself proportional to the product of the populations of susceptibles and infecteds. The term $bI(t)$ models an assumption that the rate at which infected individuals recover is proportional to the current number of infected individuals, with the period of infection being $p = 1/b$ days (see, e.g., [**2**, pp. 611–612] for a discussion of the classic SIR model).

After class, Allison, a pre-med student working hard but struggling through calculus, stayed to talk about the SIR model with me. She was troubled by the modeling assumptions implicit in the term $bI(t)$. She pointed out that the assumption of a p-day period of infection does not imply that a fraction $1/p$ of the infected population will recover each day, since there is no reason to expect a uniform distribution for the length of time that individuals in the infected class have been infected. I showed Allison the idea behind the natural discrete approximation to the continuous SIR model,

$$\begin{aligned} S_n &= S_{n-1} - aS_{n-1}I_{n-1}\Delta t, \\ I_n &= I_{n-1} + (aS_{n-1}I_{n-1} - bI_{n-1})\,\Delta t, \\ R_n &= R_{n-1} + bI_{n-1}\Delta t, \end{aligned} \tag{9}$$

obtained by applying Euler's Method to (8) with time-step Δt. Let $S(0) = S_0$, $I(0) = I_0$, and $R(0) = R_0$, so that S_n, I_n, and R_n represent the number of susceptibles, infecteds, and recovereds at time $n\Delta t$.

Allison and I decided to replace $bI_{n-1}\Delta t$ in (9) by an expression representing the actual number of infected individuals who fell sick p days before time $(n-1)\Delta t$. These individuals recover between times $(n-1)\Delta t$ and $n\Delta t$ and so move into the class of recovereds at time $n\Delta t$. Let $L = p/\Delta t$ be the number of time steps that a newly infected individual remains infected. Assume that Δt is chosen such that L is integer-valued. Then the number of infected individuals who fell sick p days before time $(n-1)\Delta t$ is given by $S_{n-1-L} - S_{n-L}$, the difference between the number of susceptible individuals at time $(n-1)\Delta t - p$ and at time $n\Delta t - p$. Our modified discrete SIR model became

$$\begin{aligned} S_n &= S_{n-1} - aS_{n-1}I_{n-1}\Delta t, \\ I_n &= I_{n-1} + aS_{n-1}I_{n-1}\Delta t - [S_{n-1-L} - S_{n-L}], \\ R_n &= R_{n-1} + [S_{n-1-L} - S_{n-L}], \end{aligned} \tag{10}$$

for $N > 0$. We assumed that we had a "history" for the epidemic, a specification of S_k, I_k, R_k for all integer k for which $-L \le k \le 0$. This modified discrete SIR model (10) takes into account the "age" structure of the population of infected individuals and is just as easy to simulate as the discrete model (9).

Let $a = 0.00025$, $p = 3$, and $\Delta t = 0.1$. Then $b = 1/3$ and $L = 30$. Imagine a student body of size 10,000 returning to school from the summer break. Suppose that one student became infected with a highly contagious flu one day before returning. Figures 1 and 2 illustrate the predictions of models (9) and (10), respectively, over the first two weeks of the school year. The modified SIR model indicates a much higher maximum number of infecteds but also a much faster recovery for the student body as a whole.

Through investigating the SIR model, Allison learned first-hand about mathematical modeling and discovered that mathematics plays an important role in her own field

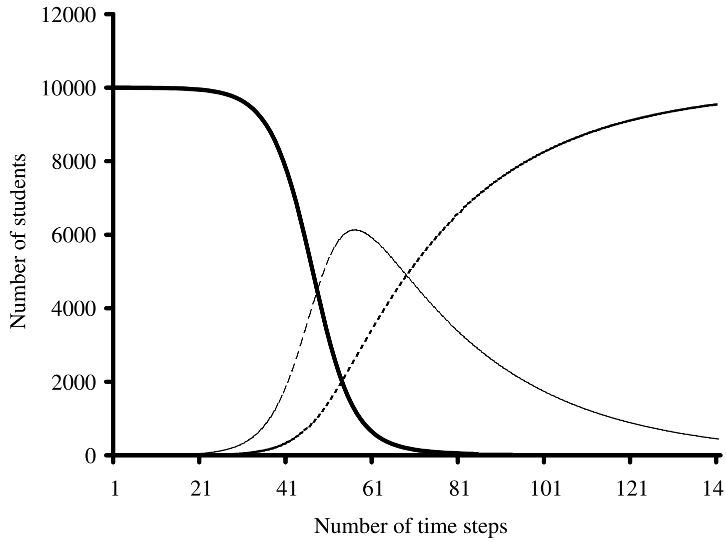

Figure 1. Classical SIR Model given by (9): An infected student returns to campus one day after infection ($a = 0.00025$, $b = 1/3$, $\Delta t = 0.1$, $S_0 = 9999$, $I_0 = 1$, $R_0 = 0$): S_n (solid), I_n (long dashes), R_n (short dashes).

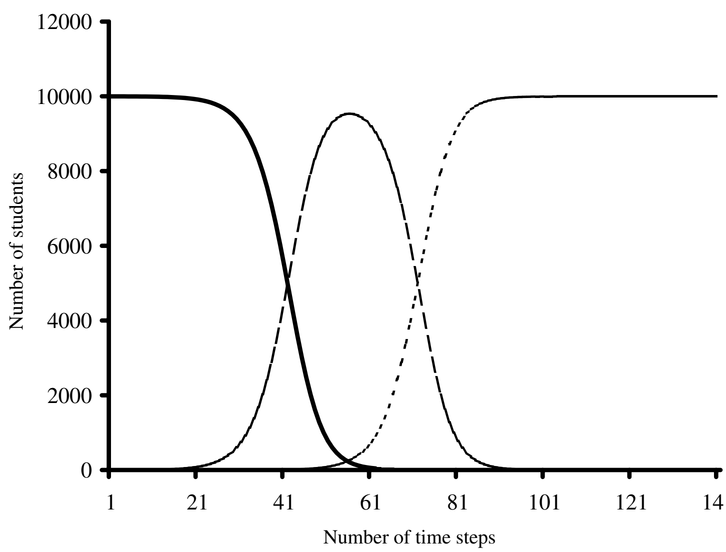

Figure 2. Modified SIR Model given by (10): An infected student returns to campus one day after infection ($a = 0.00025$, $b = 1/3$, $\Delta t = 0.1$, $S_0 = 9999$, $I_0 = 1$, $R_0 = 0$): S_n (solid), I_n (long dashes), R_n (short dashes).

of medicine. Allison was profoundly affected by the opportunity to use her knowledge and experience in order to think deeply about, and modify, an existing mathematical model. The SIR model provides an excellent opportunity to introduce calculus students to the methods and art of mathematical modeling, to analyze analytically and numerically a non-linear system of ordinary differential equations, to look at the natural connection between continuous and discrete dynamical systems, and to tie sophisticated mathematics to a subject of interest to us all, the spread of an epidemic.

Acknowledgment. The author wishes to thank University of Redlands student Allison Becker for her insight into the mathematical modeling of epidemics.

Reference

1. W. O. Kermack and A. G. McKendrick, Contributions to the mathematical theory of epidemics, *Proc. Roy. Soc.* **A 115** (1927), 700–721; **138** (1932), 55–83; **141** (1933), 92–122.
2. J. D. Murray, *Mathematical Biology*, Springer, 1993.

Maximal Revenue with Minimal Calculus

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A basic motivating example. If T is a right triangle and R is an inscribed rectangle with the right angle of T as one of its right angles, what is the maximum area of R? This problem can be solved by several standard methods. However, the quickest way is to treat the two triangular pieces of T outside of R as flaps and fold them on top of R. (See Figure 1.) The flaps will exactly cover R if and only if P is the midpoint of T's hypotenuse (since P is the midpoint of the hypotenuse if and only if $\Delta y = y$ and $\Delta x = x$). From Figure 1, we see also that Area $(R) \leq \frac{1}{2}$Area (T), with equality holding if and only if P is the midpoint of T's hypotenuse.

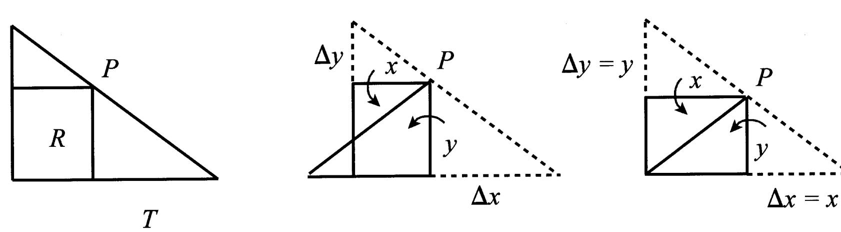

Figure 1.

The same problem in economics. Given a linear demand curve (Figure 2), what price will maximize the revenue?

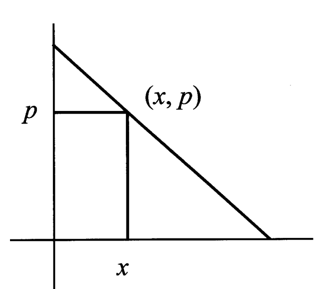

Figure 2. A linear demand curve $p = p(x)$

